UNIVERSITY OF CAMBRIDGE

MAMA/132, NST3AS/132, MAAS/132

MAT3

MATHEMATICAL TRIPOS Part III

Monday 3 June 2024 9:00 am to 11:00 am

PAPER 132

RAMSEY THEORY ON GRAPHS

Before you begin please read these instructions carefully

Candidates have TWO HOURS to complete the written examination.

Attempt no more than **THREE** questions.
There are **FOUR** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(a) Show that if G is a triangle-free graph with n vertices and maximum degree d,

$$\alpha(G) \geqslant \frac{cn \log d}{d}$$

for a constant $c > 0$. Here $\alpha(G)$ denotes the independence number of G.

(b) Let G be a graph with the property that for all $x \in V(G)$ the graph induced on the neighbourhood of x has maximum degree at most 1. That is, $\Delta(G[N(x)]) \leqslant 1$ for all $x \in V(G)$. Write $n = |V(G)|$ and $d = \Delta(G)$. Show that there is a constant $c > 0$ so that

$$\alpha(G) \geqslant \frac{cn \log d}{d}.$$

(c) For graphs H, K, let $r(H, K)$ denote the smallest n so that every red/blue colouring of $E(K_n)$ contains either a blue copy of H or a red copy of K. Let H be the graph on $\{1, \dots, 4\}$ defined by $E(H) = \{\{1,2\}, \{1,3\}, \{2,3\}, \{1,4\}, \{2,4\}\}$. Show that $r(H, K_k) \leqslant ck^2/\log k$ for some constant $c > 0$.

2

(a) For $r \geqslant 2$, define the *r-uniform hypergraph Ramsey number* $R^{(r)}(k)$. Show that there is a constant $c > 0$ for which $R^{(3)}(k) \geqslant 2^{ck^2}$.

(b) Prove that there is a constant $c > 0$ so that $R^{(3)}(k) \leqslant 2^{2^{ck}}$.

(c) Let H be the 3-uniform hypergraph with $V(H) = \{1, 2, 3, 4\}$ and $E(H) = \{\{1,2,3\}, \{1,2,4\}, \{1,3,4\}\}$. Show that there is a constant $c > 0$ so that $r(H, K_k^{(3)}) \leqslant k^{ck}$.

Recall that for any r-uniform hypergraphs H, K, $r(H, K)$ denotes the smallest n so that every red/blue colouring of $K_n^{(r)}$ either contains a blue copy of H or a red copy of K.

3

(a) For $s, k \geqslant 1$ and a graph G, define what it means for $R \subseteq V(G)$ to be (s,k)-*rich*. Now let G be a graph with $|G| = n$ and $e(G) = m$. Let $r, s, k, t \geqslant 1$ be such that

$$\frac{(2m)^t}{n^{2t-1}} - \binom{n}{s}\left(\frac{k}{n}\right)^t \geqslant r.$$

Show that there exists an (s,k)-rich set $R \subseteq V(G)$, with $|R| \geqslant r$.

(b) Let H be a bipartite graph on k vertices with maximum degree d. Let G be a graph, and let $R \subset V(G)$ be a (d,k)-rich set in G with $|R| \geqslant k$. Show that $H \subset G$.

(c) Let H be a bipartite graph on k vertices with maximum degree d. Using the above, show that $r(H) \leqslant k^{1+\varepsilon}$ for every $\varepsilon > 0$, provided k is sufficiently large depending on d and ε.

4

(a) State the *regularity lemma*. Be sure to include the definition of an ε-*uniform pair*.

(b) Using the regularity lemma, show that if H is a graph with chromatic number 3, then

$$\lim_{n\to\infty} \frac{ex(n,H)}{\binom{n}{2}} = \frac{1}{2}.$$

(c) For graphs G, H, we write $G \to H$ if every red/blue colouring of the edges of G contains a monochromatic copy of H. Let $G \sim G(n, 1/2)$. Show that

$$\lim_{n\to\infty} \mathbb{P}(G \to C_5) = 1.$$

[*For any of the parts you may use Ramsey's theorem, Turán's theorem or the Chernoff inequality without proof, so long as they are stated correctly.*]

END OF PAPER