UNIVERSITY OF CAMBRIDGE

MAMA/129, NST3AS/129, MAAS/129

MAT3

MATHEMATICAL TRIPOS Part III

Tuesday 4 June 2024 9:00 am to 11:00 am

PAPER 129

INTRODUCTION TO ADDITIVE COMBINATORICS

Before you begin please read these instructions carefully

Candidates have TWO HOURS to complete the written examination.

Attempt no more than **THREE** questions.
There are **FOUR** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(i) State and prove Plünnecke's inequality.

Let G be an abelian group.

(ii) Given finite sets $A, B \subseteq G$, show that there exists a set $X \subseteq B$ of size $|X| \leqslant 2|A+B|/|A| - 1$ such that for every $b \in B$, there are more than $|A|/2$ triples $(x, a, a') \in X \times A \times A$ such that $b = a - a' + x$.

Deduce that $B - B \subseteq A - A + X - X$.

(iii) Given a real number $K \geqslant 1$, we say that a set $H \subseteq G$ is a K-*approximate group* if it is symmetric (i.e. $H = -H$), contains the additive identity, and $H + H$ can be covered by at most K translates of H.

Show that if $|A + A| \leqslant K|A|$, then there exists a constant $C > 0$ and a CK^C-approximate group $H \subseteq G$ such that $|H| \leqslant CK^C|A|$ and $A \subseteq H + x$ for all $x \in A$.

2

Let $A \subseteq \mathbb{F}_2^n$ be a set of density $\alpha > 0$, and let $\beta > 0$ denote the density of $B = \mathbb{F}_2^n \setminus (A + A)$.

(i) Define $\operatorname{Spec}_\rho(1_A)$ for $\rho > 0$, and state Chang's theorem.

(ii) By considering the inner product $\langle 1_A * 1_A, 1_B \rangle$ or otherwise, show that there is a subspace $V \leqslant \mathbb{F}_2^n$ of codimension $O(\alpha^{-2}\log(\beta^{-1}))$ such that

$$\sum_{t \in V^\perp} |\widehat{1_A}(t)|^2 \geqslant 3\alpha^2/2.$$

(iii) Let $1 \leqslant k \leqslant n$ be an integer, and let $H \leqslant \mathbb{F}_2^n$ be a subspace of dimension k. By considering $\mathbb{E}_x 1_B * \mu_H(x)$ or otherwise, show that if $\beta < 2^{-k}$, then $A + A$ contains a coset of H.

(iv) Deduce that if $1 \leqslant k \leqslant n$, then either $A + A$ contains a coset of a subspace H of dimension k, or there exists a subspace V of codimension $O(\alpha^{-2}k)$ such that

$$\|1_A * \mu_V\|_\infty \geqslant 3\alpha/2.$$

(v) Conclude that there exists a constant $C > 0$ such that $A + A$ contains a coset of a subspace of dimension at least $C\alpha^2 n$.

3

Let G be a finite abelian group. For $x \in G$, let τ_x denote the shift-by-x operator. That is, for any function $g : G \to \mathbb{C}$, $\tau_x g(y) = g(y + x)$ for all $y \in G$.

(i) State Croot and Sisask's almost-periodicity result.

Let $A \subseteq G$ be a subset of density $\alpha > 0$.

(ii) Show that for all $\epsilon > 0$ and all integers $k \geqslant 1$ and $p \geqslant 2$, there is a set X of size $|X| \geqslant \alpha^{O(\epsilon^{-2}k^2 p)}|G|$ such that

$$\|\tau_x(1_{-A} * \mu_A) - 1_{-A} * \mu_A\|_p \leqslant \epsilon$$

for all $x \in kX - kX$.

(iii) Suppose instead that there is a set Y of size $|Y| \geqslant \alpha^{O(\epsilon^{-2}k^2)}|G|$ such that

$$\|\tau_y(1_{-A} * \mu_A) - 1_{-A} * \mu_A\|_\infty \leqslant \epsilon$$

for all $y \in kY - kY$. Show that

$$\|1_{-A} * \mu_A * \mu - 1_{-A} * \mu_A\|_\infty \leqslant \epsilon,$$

where $\mu = \mu_Y^{(k)} * \mu_{-Y}^{(k)}$ and $\mu_Y^{(k)}$ denotes the k-fold convolution of μ_Y with itself.

(iv) Let Y and μ be as in (iii). By making appropriate choices of ϵ and k, show that there exists a Bohr set $B = B(\Lambda, 1/(6|\Lambda|))$ with $\Lambda \subseteq \widehat{G}$ a dissociated set of size $|\Lambda| = O(\log(\alpha^{-1}))$ such that for any $y \in B$ and any $x \in G$,

$$|1_{-A} * \mu_A * \mu(x + y) - 1_{-A} * \mu_A * \mu(x)| \leqslant 1/3.$$

Conclude that for any $y \in B$ and any $x \in G$,

$$|1_{-A} * \mu_A(x + y) - 1_{-A} * \mu_A(x)| \leqslant 2/3.$$

By making an appropriate choice of x, deduce that $A - A$ contains B.

4

(i) State Szemerédi's theorem on 4-term arithmetic progression in subsets of $\mathbb{F}_p^n$ with $p \geqslant 5$.

(ii) Outline why the Fourier-analytic method used to prove Meshulam's theorem does not suffice to obtain Szemerédi's theorem. [You do not need to explain how to overcome this obstacle, or perform any detailed calculations.]

Let $p \geqslant 5$ be a prime, let $A \subseteq \mathbb{F}_p^n$ and let $\eta > 0$.

(iii) Show that if A contains at least $\eta|A|^2$ 3-term arithmetic progressions, then it contains at least $\eta^2|A|^3$ additive quadruples.

(iv) Show that there exists a constant $C = C(\eta) \geqslant 1$ such that if $A \subseteq \mathbb{F}_p^n$ has size $|A| \geqslant C$ and contains at least $\eta|A|^2$ 3-term arithmetic progressions, then A contains a non-trivial 4-term arithmetic progression.

[You may use any theorems from lectures without proof provided they are stated clearly.]

END OF PAPER