MAMA/128, NST3AS/128, MAAS/128

MAT3

MATHEMATICAL TRIPOS Part III

Thursday 6 June 2024 1:30 pm to 4:30 pm

PAPER 128

FORCING AND THE CONTINUUM HYPOTHESIS

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

Attempt **ALL** questions.
There are **FOUR** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(a) Define operations $\mathcal{F}_1$, $\mathcal{F}_2$ and $\mathcal{F}_3$ as follows: $\mathcal{F}_1(x,y) = \{x,y\}$, $\mathcal{F}_2(x,y) = \bigcup x$, $\mathcal{F}_3(x,y) = x \setminus y$. Explicitly find a term $\mathcal{G}$ built from the symbols $\mathcal{F}_1, \mathcal{F}_2$ and $\mathcal{F}_3$ such that for any sets a, b and c,

$$\mathcal{G}(a,b,c) = (a \cup \{b\}) \cap c.$$

(b) Show that the constructible universe L satisfies Replacement in the following form:

For every formula φ,
$\forall a\, \forall u\, (\forall x \in a\, \exists! y\, \varphi(x,y,u) \to \exists b\, \forall x \in a\, \exists y \in b\, \varphi(x,y,u))$.

(c) Suppose that $\mathrm{V} = \mathrm{L}$. Show that $\mathrm{L}_{\omega_1} = \mathrm{H}_{\aleph_1}$.

[*You may use without proof that for every ordinal* α, L_α *is a transitive set and* $|\mathrm{L}_\alpha| = |\alpha|$.]

2

(a) Fix sets I and J and a regular cardinal κ. Give definitions of the following partial orders, explicitly defining the set, the order, and the maximal element.

(i) $\mathrm{Fn}(I,J)$,

(ii) $\mathrm{Fn}_\kappa(I,J)$.

(b) Let J be a countable set and I a non-empty set. Show that $\mathrm{Fn}(I,J)$ has the countable chain condition.

(c) Let M be a countable transitive model of ZFC, let $\mathbb{P} = \mathrm{Fn}(\omega, 2)$ and let G by $\mathbb{P}$-generic over M. Let $c = \bigcup G$ and let $d\colon \omega \to 2$ be the function defined by $d(n) = c(2n)$. Show that there is some $\mathbb{P}$-generic filter H over M such that $d = \bigcup H$.

3

(a) State the forcing theorem.

(b) Suppose that the forcing theorem holds for a formula φ. Prove that the forcing theorem holds for the formula $\exists x\varphi(x)$.

(c) Call a function $f\colon \omega_1 \to \omega_1$ *normal on* ω_1 if it is a total, strictly increasing function and for all limit ordinals $\delta \in \omega_1$, $f(\delta) = \sup\{f(\alpha) : \alpha \in \delta\}$. Let $\mathbb{B}$ be the partial order whose conditions are finite partial functions $p\colon \omega_1 \to \omega_1$ such that there is a normal function $f\colon \omega_1 \to \omega_1$ with $p \subseteq f$. Let the ordering on $\mathbb{B}$ be given by reverse inclusion, that is $q \leqslant p$ iff $q \supseteq p$.

Suppose that M is a countable transitive model of ZFC and let G be $\mathbb{B}$-generic over M. Show that, in $M[G]$, $F = \bigcup G$ is a normal function on $(\omega_1)^M$.

[A function is *total* if it is defined on every element of the domain.]

4

(a) State the Reflection Theorem.

(b) Suppose that M is a countable transitive model of ZFC + V = L and let G be a $\mathrm{Fn}(\omega, \omega_1)$-generic filter over M. Give explicit examples of formulas φ_1, φ_2 such that

(i) φ_1 is upwards absolute but not downwards absolute between M and $M[G]$,

(ii) φ_2 is downwards absolute but not upwards absolute between M and $M[G]$.

You should give a full argument for why your chosen formulas are downwards and upwards absolute respectively.

[*You may assume that the following terms are absolute: function, injection, surjection, $x = dom(f)$, $x = ran(f)$, α is an ordinal, $x = \omega$.*]

(c) Show that for any formula φ the following are equivalent:

(i) φ is Δ_1^{ZF},

(ii) For some finite subset T of axioms of ZF, ZF proves that φ is absolute for any transitive class M such that $M \models \psi$ for all $\psi \in T$.

END OF PAPER