MAMA/126, NST3AS/126, MAAS/126

MAT3

MATHEMATICAL TRIPOS Part III

Tuesday 4 June 2024 9:00 am to 12:00 pm

PAPER 126

ABELIAN VARIETIES

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

Attempt no more than **THREE** questions.
There are **FOUR** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 (a) Let $A \to B$ be a ring homomorphism. Define the module $\Omega_{B/A}$ of differentials.

Let $A \to B \to C$ be ring homomorphisms. Write down the first exact sequence of differentials for $C/B/A$.

Suppose $B \to C$ is surjective, with kernel I. Write down the second exact sequence for $C/B/A$.

Let L/K be a finite extension of fields. Show that $\Omega_{L/K} = 0$ if and only if L/K is separable.

Suppose that $\mathrm{char}(K) = p > 0$. Compute $\Omega_{L/K}$ when:

$$L = K(a), \text{ where } a^p \notin K,\ a^{p^2} \in K$$
$$L = K(a, b), \text{ where } a \notin K,\ b \notin K(a) \text{ and } a^p,\ b^p \in K.$$

Let k be a field. Compute $\Omega_{X/Y}$ for each of the following morphisms $f \colon X \to Y$:

(i) $X = \operatorname{Spec} k[x, y]/(y^2 - x) \to Y = \operatorname{Spec} k[t]$, $f^\#(t) = x$

(ii) $X = \operatorname{Spec} k[x] \to Y = \operatorname{Spec} k[u, v]/(u^3 - v^2)$, $f^\#(u) = x^2$, $f^\#(v) = x^3$

(iii) $X = \operatorname{Spec} k[x, y]/(xy) \to Y = \operatorname{Spec} k[t]$, $f^\#(t) = x$

and in each case write down the support of $\Omega_{X/Y}$.

(b) Let $f \colon X \to Y$ be a morphism of schemes. What does it mean to say that f is flat?

Which of the morphisms (i)–(iii) are flat? Justify your answers.

2 In this question, all schemes are assumed to be separated and Noetherian.

(i) Let $f\colon X \to Y$ be an affine morphism of schemes, and $\mathcal{F}$ a quasicoherent $\mathcal{O}_X$-module. Show that for every $p \geqslant 0$, $H^p(X, \mathcal{F}) = H^p(Y, f_*\mathcal{F})$.

(ii) State the Mayer–Vietoris theorem in sheaf cohomology.

Let $X = \bigcup_{i=1}^m U_i$ be an open cover of a scheme X and $\mathcal{F}$ an abelian sheaf on X. Suppose that for every nonempty subset $\{i_1, \ldots, i_r\} \subset \{1, \ldots, m\}$ and every $p \geqslant 0$, $H^p(U_{i_1} \cap \cdots \cap U_{i_r}, \mathcal{F}) = 0$. Show that for every $p \geqslant 0$, $H^p(X, \mathcal{F}) = 0$.

(iii) Let A be a reduced Noetherian ring, and $f\colon X \to \operatorname{Spec} A$ a proper morphism. Suppose that $\mathcal{F}$ is a coherent $\mathcal{O}_X$-module, flat over $\operatorname{Spec} A$, and $n \geqslant 0$ an integer such that $H^p(X, \mathcal{F}) = 0$ for every $p > n$.

State a theorem about the existence of a complex of locally free A-modules computing the cohomology of $\mathcal{F}$. Use it to show that $H^n(X, \mathcal{F}) = 0$ if and only if, for every $s \in \operatorname{Spec} A$, $H^n(X_s, \mathcal{F}(s)) = 0$.

(iv) Let $f\colon X \to Y$ be a proper morphism of schemes, with Y reduced, and $\mathcal{F}$ a coherent $\mathcal{O}_X$-module, flat over Y. Suppose that for every $y \in Y$ and every $p \geqslant 0$, $H^p(X_y, \mathcal{F}(y)) = 0$. Show that for every $p \geqslant 0$, $H^p(X, \mathcal{F}) = 0$.

3 (a) What is a *group scheme* over a field k?

Let G and H be group schemes over k. What is a *homomorphism of group schemes* from H to G?

Suppose that G and H are commutative. Let $\operatorname{Hom}_k(H, G)$ be the set of homomorphisms from H to G, and for any k-scheme S, let $G(S)$ be the group of S-valued points of G.

Show that $\operatorname{Hom}_k(H, G)$ is a subgroup of the group $G(H)$, and that for every k-algebra R, every f, $g \in \operatorname{Hom}_k(H, G)$ and every $x \in H(R)$, $f_R(x) + g_R(x) = (f + g)_R(x)$. Show also that for every integer $n \geqslant 1$, $nf = f \circ [n]_H = [n]_G \circ f$, where $[n]_G$, $[n]_H$ are the multiplication-by-n morphisms on G and H.

(b) State a version of Mumford's Rigidity Lemma.

Let k be an algebraically closed field. Suppose that X and Y are abelian varieties over k, and that S is a k-variety.

Let $f\colon X \times S \to Y$ be a morphism such that $f(e, s) = e$ for every $s \in S(k)$. Show that there exists $g \in \operatorname{Hom}_k(X, Y)$ such that for every $s \in S(k)$, $f|_{X \times s} = g$.

4 Let X be an abelian variety over an algebraically closed field k.

(i) State the Theorem of the Square.

(ii) Define the map $\phi_{\mathcal{L}}\colon X(k) \to \operatorname{Pic}(X)$ for a line bundle $\mathcal{L}$ on X, and show that it is a homomorphism.

Show that $\phi_{\mathcal{L}\otimes\mathcal{M}}(x) = \phi_{\mathcal{L}}(x)\phi_{\mathcal{M}}(x)$ and $\phi_{\mathcal{L}^{\otimes n}}(x) = \phi_{\mathcal{L}}(nx)$ for any $\mathcal{L}, \mathcal{M} \in \operatorname{Pic}(X)$, $x \in X(k)$, and $n \geqslant 1$.

Deduce that the quotient $\operatorname{Pic}(X)/\operatorname{Pic}^0(X)$ is torsion-free, where $\operatorname{Pic}^0(X) = \{\mathcal{L} \in \operatorname{Pic}(X) \mid \phi_{\mathcal{L}} \text{ is trivial}\}$, and that for every $\mathcal{L} \in \operatorname{Pic}(X)$, $\operatorname{im}(\phi_{\mathcal{L}}) \subset \operatorname{Pic}^0(X)$.

(iii) Let $\Lambda(\mathcal{L}) = m^*\mathcal{L} \otimes pr_1^*\mathcal{L}^\vee \otimes pr_2^*\mathcal{L}^\vee \in \operatorname{Pic}(X \times X)$. Show that $\mathcal{L} \in \operatorname{Pic}^0(X)$ if and only if $\Lambda(\mathcal{L})$ is trivial.

(iv) Let Y be any k-variety, and $f,\ g\colon Y \to X$ morphisms. Let $f + g$ denote their sum in the group $X(Y)$.

Show that if $\mathcal{L} \in \operatorname{Pic}^0(X)$ then $(f+g)^*\mathcal{L} \simeq f^*\mathcal{L} \otimes g^*\mathcal{L}$ in $\operatorname{Pic}(Y)$. Deduce that if $\mathcal{L} \in \operatorname{Pic}^0(X)$, then $i^*\mathcal{L} \simeq \mathcal{L}^\vee$ and for every $n \in \mathbb{Z}$, $[n]^*\mathcal{L} \simeq \mathcal{L}^{\otimes n}$.

By considering $\mathcal{L}^{\otimes 2}$, show that if $i^*\mathcal{L} \simeq \mathcal{L}^\vee$ then $\mathcal{L} \in \operatorname{Pic}^0(X)$.

END OF PAPER