UNIVERSITY OF CAMBRIDGE

MAMA/123, NST3AS/123, MAAS/123

MAT3

MATHEMATICAL TRIPOS Part III

Thursday 6 June 2024 9:00 am to 12:00 pm

PAPER 123

ALGEBRAIC NUMBER THEORY

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

Attempt no more than **FOUR** questions.
There are **FIVE** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 This question concerns the *ideal theoretic* version of global class field theory. Let K be a number field.

a) Give the definition of a *modulus* of K.

b) Let $\mathfrak{m}$ be a modulus of K. Define the ray class group mod $\mathfrak{m}$, defining each object in the definition.

c) Give the formula for the order of the ray class group mod $\mathfrak{m}$.

d) Let $K = \mathbb{Q}(\sqrt{-7})$. Show using the Artin map that 5 is inert in K.

e) Again let $K = \mathbb{Q}(\sqrt{-7})$ and let $\mathfrak{m} = 5\mathcal{O}_K$ be a modulus of K. Show that the ray class field mod $\mathfrak{m}$ is an extension of degree 12 over K. You may use that the ideal class group of K is trivial.

2 This question concerns the *ideal theoretic* version of global class field theory. Let K be a number field and $\mathfrak{m}$ a modulus.

a) Let L/K be an abelian extension and let $\mathfrak{p} \subset \mathcal{O}_K$ be an unramifed prime ideal. Define the Artin symbol $\left(\frac{L/K}{\mathfrak{p}}\right)$.

b) Use the Artin map of $\mathbb{Q}(\zeta_8)/\mathbb{Q}$ to construct isomorphisms

$$I_{\mathbb{Q}}(8\infty)/P_{\mathbb{Q}}(8\infty) \cong \mathrm{Gal}(\mathbb{Q}(\zeta_8)/\mathbb{Q}) \cong (\mathbb{Z}/8\mathbb{Z})^{\times}$$

and deduce that

$$I_{\mathbb{Q}}(8\infty)/P_{\mathbb{Q}}(8\infty) = \{[\mathbb{Z}], [3\mathbb{Z}], [5\mathbb{Z}], [7\mathbb{Z}]\}$$

Identify the element of $(\mathbb{Z}/8\mathbb{Z})^{\times}$ corresponding to complex conjugation $c \in \mathrm{Gal}(\mathbb{Q}(\zeta_8)/\mathbb{Q})$ under the usual isomorphism $\mathrm{Gal}(\mathbb{Q}(\zeta_8)/\mathbb{Q}) \cong (\mathbb{Z}/8\mathbb{Z})^{\times}$.

c) Let $H = \{\mathbb{Z}, 7\mathbb{Z}\}P_{\mathbb{Q}}(8\infty)$ and show this contains $P_{\mathbb{Q}}(8\infty)$ as a subgroup of index 2.

d) State the (**ideal theoretic**) Existence Theorem.

e) Show that via the Existence Theorem the group H corresponds to $\mathbb{Q}(\sqrt{2})$.

3 In this question we let K be a number field and we let m be a positive integer.

a) Give the definition of a Dirichlet character mod m, and show how one can extend such a character to define a L-series.

b) Define the Dedekind zeta function of K.

c) Let K be any abelian extension of $\mathbb{Q}$. Demonstrate explicitly that characters of the Galois group can be considered as Dirichlet characters and write $\zeta_K(s)$ as a product of Dirichlet L-series.

d) How many Dirichlet characters are there mod 7? Write them all out.

e) Next let $K = \mathbb{Q}(\sqrt{-7})$. Which of the characters mod 7 gives the unique character corresponding to K?

f) Again let $K = \mathbb{Q}(\sqrt{-7})$. Use the identities below to prove the class number h_K of K equals 1. If χ is a primitive character mod $m \geqslant 3$, then

$$|L(\chi,1)| = \begin{cases} \frac{2}{\sqrt{m}} \left| \sum_{\substack{k \in (\mathbb{Z}/m\mathbb{Z})^\times \\ k<m/2}} \chi(k) \log(\sin(\pi k/m)) \right|, & \text{if } \chi(-1) = 1, \\ \frac{\pi}{|2-\chi(2)|\sqrt{m}} \left| \sum_{\substack{k \in (\mathbb{Z}/m\mathbb{Z})^\times \\ k<m/2}} \chi(k) \right|, & \text{if } \chi(-1) = -1. \end{cases}$$

You may use the analytic class number formula provided you state it precisely.

4 This question concerns the *idele theoretic* version of global class field theory. Let K be a number field, and let $\mathfrak{m}$ be a modulus.

a) Give the definition of the idele group $\mathbb{I}_K$.

b) For any $\alpha \in \mathbb{I}_K$, explain what it means when $\alpha \equiv 1 \mod \mathfrak{m}$, and define the group $\mathbb{I}_K(\mathfrak{m})$. Define the congruence subgroup mod $\mathfrak{m}$ of the idele class group C_K.

c) Let m be a positive integer and ∞ the infinite prime of $\mathbb{Q}$. Describe $\mathbb{I}_{\mathbb{Q}}(\mathfrak{m})$ for $\mathfrak{m} = m\infty$.

d) Show the index $[C_{\mathbb{Q}} : C_{\mathbb{Q}}(\mathfrak{m})]$ is equal to the Euler totient function $\phi(m) = m\prod_{p|m}(1-\frac{1}{p})$. [*Hint:* $[U_p : U_p^{n_p}] = p^{n_p-1} \cdot (p-1)$ *if* $n_p > 0$.]

e) Assuming $C_{\mathbb{Q}}(\mathfrak{m}) \subset N_{\mathbb{Q}(\zeta_m)/\mathbb{Q}} C_{\mathbb{Q}(\zeta_m)}$, for ζ_m a primitive m-th root of unity, use the above to show that $C_{\mathbb{Q}}(\mathfrak{m})$ is the norm group of the cyclotomic field $\mathbb{Q}(\zeta_m)$.

f) State the Kronecker–Weber Theorem and deduce it from the main results of global class field theory.

5 This question is about density. Let K be a number field.

a) Define Dirichlet density.

b) Let $\mathcal{C}$ be a fixed class in the ideal class group. Prove that the set of prime ideals $\mathfrak{p} \in \mathcal{C}$ has Dirichlet density $1/h_K$. You may assume that $L(\chi, 1) \neq 0$ for the appropriate non–trivial characters.

c) State the Chebotarev density theorem.

d) Compute the Hilbert class field of $K = \mathbb{Q}(\sqrt{26})$. You may use it has class number 2.

e) Use the previous two parts to compute the density of the principal prime ideals of $K = \mathbb{Q}(\sqrt{26})$.

END OF PAPER