UNIVERSITY OF CAMBRIDGE

MAMA/120

MAT3

MATHEMATICAL TRIPOS Part III

Thursday 30 May 2024 9:00 am to 12:00 pm

PAPER 120

MODEL THEORY AND NON-CLASSICAL LOGIC

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

Attempt **ALL** questions.
There are **THREE** questions in total.
Questions 1 and 3 carry 30 marks each.
Question 2 carries 40 marks.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(a) Let $\mathcal{L}$ be a first-order language and $\mathcal{M}$ be an $\mathcal{L}$-structure. Define the *diagram* and the *elementary diagram* of $\mathcal{M}$. Describe the technique known as *the method of diagrams.*

Recall that a family of $\mathcal{L}$-structures $\{\mathcal{A}_i \mid i \in I\}$ is an *embedding chain* (respectively, an *elementary embedding chain*) if the set I is a total order and $i \leqslant j$ implies that $\mathcal{A}_i$ is a substructure (respectively, an elementary substructure) of $\mathcal{A}_j$.

(b) Suppose that $\{\mathcal{A}_i \mid i \in I\}$ is an embedding chain of $\mathcal{L}$-structures. Show that $\mathcal{A} := \bigcup_{i \in I} \mathcal{A}_i$ can be made into an $\mathcal{L}$-structure such that $\mathcal{A}_i$ is a substructure of $\mathcal{A}$ for all $i \in I$. Furthermore, show that if the chain is an elementary embedding chain, then each $\mathcal{A}_i$ is an elementary substructure of $\mathcal{A}$.

[*For item (b), you may assume that quantifier-free formulae are preserved under extensions/substructures without proof.*]

(c) A $\forall\exists$-sentence is one of the form $\forall \bar{x}.\ \exists \bar{y}.\ \phi(\bar{x}, \bar{y})$, where ϕ is quantifier-free. Show that a first-order theory $\mathcal{T}$ admits an axiomatisation by $\forall\exists$-sentences if and only if it is preserved under unions of embedding chains of structures (i.e., if all structures in an embedding chain model $\mathcal{T}$, then their union also models $\mathcal{T}$).

[*Hint: consider the set $\mathcal{T}_0$ of all $\forall\exists$-sentences provable from $\mathcal{T}$.*]

(d) What does it mean for a theory $\mathcal{T}$ to be *model-complete*? Prove that a model-complete theory must be axiomatisable by $\forall\exists$-sentences.

2

(a) Let $\mathcal{M}$ be an $\mathcal{L}$-structure, X be a subset of $\mathcal{M}$, and n be a natural number. Define what is meant by a *complete n-type* of $\mathcal{M}$ over X and define the topological space $S_n^{\mathcal{M}}(X)$. What is an *isolated n-type*?

(b) Let p be an n-type of $\mathcal{M}$ over $X \subseteq \mathcal{M}$. Show that there is an elementary extension $\mathcal{N}$ of $\mathcal{M}$ such that p is realised in $\mathcal{N}$.

(c) Show that $S_n^{\mathcal{M}}(X)$ is totally disconnected, i.e., for any distinct points $p, q \in S_n^{\mathcal{M}}(X)$, there are disjoint open sets U, V such that $p \in U$, $q \in V$, and $S_n^{\mathcal{M}}(X) = U \cup V$.

(d) Let $\mathcal{T}$ be a Skolem $\mathcal{L}$-theory, $\mathcal{M} := F(\eta)$ be an Ehrenfeucht-Mostowski model of $\mathcal{T}$, and X be a subset of $\mathcal{M}$. Show that if η is a well-ordering, then the number of complete 1-types over X realised in $\mathcal{M}$ is at most $|\mathcal{L}| + |X|$.

(e) A model of a theory $\mathcal{T}$ is a *prime model* if it elementarily embeds into any model of $\mathcal{T}$. Let $\mathcal{L}$ be a countable language and let $\mathcal{T}$ be a complete $\mathcal{L}$-theory with infinite models. Show that $\mathcal{M} \models \mathcal{T}$ is a prime model if and only if it is countable and $\mathrm{tp}^{\mathcal{M}}(\bar{a}/\emptyset)$ is isolated for every natural number n and tuple $\bar{a} \in \mathcal{M}^n$.

[*For item (e), you may use any results proved in the lectures provided that you state them correctly and precisely.*]

(f) Let $\mathcal{L}$ be a countable language and let $\mathcal{T}$ be a complete $\mathcal{L}$-theory with infinite models. Prove that if $\mathcal{T}$ admits a prime model, then the set of isolated points of $S_n(\mathcal{T})$ is a dense subset of $S_n(\mathcal{T})$ for all n.

3 (a) State and prove the *Curry-Howard correspondence* between the implicational fragments of the simply typed λ-calculus and the intuitionistic propositional calculus.

(b) Let σ and τ be type variables. Show that there is no simply typed $\lambda(\to)$-term M such that $\Vdash M \colon ((\sigma \to \tau) \to \sigma) \to \sigma$.

(c) Is there a first-order theory $\mathcal{T}$ on some language $\mathcal{L}$ that axiomatises those Heyting algebras H for which $\neg\neg p = p$ for only finitely many $p \in H$? Justify your claim.

[*For items (b) and (c), you may use any results proved in the lectures provided that you state them correctly and precisely.*]

END OF PAPER