UNIVERSITY OF CAMBRIDGE

MAMA/119

MAT3

**MATHEMATICAL TRIPOS Part III**

Thursday 30 May 2024 1:30 pm to 4:30 pm

# PAPER 119

# CATEGORY THEORY

**Before you begin please read these instructions carefully**

Candidates have THREE HOURS to complete the written examination.

Attempt no more than **FOUR** questions.
There are **SIX** questions in total.
The questions carry equal weight.

**STATIONERY REQUIREMENTS**
Cover sheet
Treasury tag
Script paper
Rough paper

**SPECIAL REQUIREMENTS**
None

**You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.**

**1** State the Yoneda Lemma, and deduce that if $\mathcal{C}$ is a small category then any functor $F\colon \mathcal{C} \to \mathbf{Set}$ is the codomain of an epimorphism from a coproduct of representable functors.

A functor $F\colon \mathcal{C} \to \mathcal{D}$ is called a *discrete fibration* if, whenever we are given $A \in \mathrm{ob}\,\mathcal{C}$ and a morphism $f\colon B \to FA$ in $\mathcal{D}$, there exists a unique morphism $\widetilde{f}$ in $\mathcal{C}$ with $\mathrm{cod}\,\widetilde{f} = A$ and $F\widetilde{f} = f$. Show that a discrete fibration preserves monomorphisms. Show also that, for any functor $F : \mathcal{C} \to \mathcal{D}$ and any $B \in \mathrm{ob}\,\mathcal{D}$, the forgetful functor $(F \downarrow B) \to \mathcal{C}$ is a discrete fibration.

Let $\mathcal{C}$ be a small category. Show that the following three conditions are equivalent:

(i) Every morphism of $\mathcal{C}$ is a monomorphism.

(ii) For every representable $F : \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$, $(1 \downarrow F)$ is a preorder, where 1 denotes a one-element set.

(iii) There exists a preorder $\mathcal{P}$ and a discrete fibration $\mathcal{P} \to \mathcal{C}$ which is surjective on objects.

**2** Explain what is meant by the *limit* of a diagram, and prove that all small limits may be constructed from products and equalizers.

Let **Fld** be the category of fields, considered as a full subcategory of the category **CRng** of commutative rings. Show that the limit in **CRng** of any (small) connected diagram in **Fld** is a field, and deduce that **Fld** has all such limits. [*You may assume that the forgetful functor* **CRng** $\to$ **Set** *creates limits.*] Does **Fld** have limits of shape $J$ for any disconnected category $J$? Justify your answer.

A (possibly empty) family of cones under a diagram $D$ is called a *multicolimit* for $D$ if it contains one initial object for each connected component of the category of cones under $D$. Show that every small diagram in **Fld** has a multicolimit. [*Hint: consider the diagram formed by the apexes of the cones in a connected componenet.*]

**3** State a form of the Adjoint Functor Theorem applicable to partially ordered sets, and use it to prove that the category **CSLat** of complete semilattices (posets with arbitrary joins) and maps preserving arbitrary joins is self-dual, i.e. there exists a functor $(-)^*\colon \mathbf{CSLat}^{\mathrm{op}} \to \mathbf{CSLat}$ whose square is the identity.

Show also that **CSLat** is enriched over itself, i.e. that there exists a functor

$$(A,B) \mapsto [A,B]\colon \mathbf{CSLat}^{\mathrm{op}} \times \mathbf{CSLat} \to \mathbf{CSLat}$$

whose composite with the forgetful functor **CSLat** $\to$ **Set** is the functor $\mathbf{CSLat}\,(-,-)$. Show that the self-duality is enriched in the sense that $[B^*, A^*] \cong [A,B]$. Hence show that $A^* \cong [A,2]$ where 2 denotes the two-element lattice, and deduce that $[A,[B,C]] \cong [B,[A,C]]$ for all $A$, $B$ and $C$. [*You may assume that* $C$ *may be expressed as the limit of a diagram whose vertices are copies of* 2.]

**4** Explain what is meant by a *monadic* adjunction, and state the Crude Monadicity Theorem.

For a monad $\mathbb{T} = (T, \eta, \mu)$ on **Set**, show that the following are equivalent:

(i) The free functor $F\colon \mathbf{Set} \to \mathbf{Set}^{\mathbb{T}}$ reflects isomorphisms.

(ii) $F$ is faithful.

(iii) $\eta$ is (pointwise) monic.

(iv) $\eta_2$ is monic, where $2 = \{0, 1\}$.

(v) There exists a $\mathbb{T}$-algebra with more than one element.

[*Standard results about adjunctions may be quoted without proof.*]

If these conditions are satisfied, and additionally $T$ preserves finite coproducts, show that the adjunction $(\mathbf{Set} \rightleftarrows \mathbf{Set}^{\mathbb{T}})$ is comonadic. [*Hint: first show that any coreflexive equalizer diagram $A \rightarrowtail B \rightrightarrows C$ in* **Set***, where $A \neq \emptyset$, can be given the structure of a split equalizer diagram.*]

**5** Explain what is meant by an *enriched category* over a symmetric monoidal category, and by the *underlying ordinary category* of an enriched category. Show that any closed symmetric monoidal category has a canonical enrichment over itself.

Show that the category **Poset** of partially ordered sets is cartesian closed.

Given morphisms $f\colon A \to B$ and $g\colon B \to A$ in (the underlying ordinary category of) a **Poset**-enriched category $\mathcal{C}$, we say $f$ is *left adjoint* to $g$ if $fg \leqslant 1_B$ and $1_A \leqslant gf$. Show that the left adjoints form a subcategory of $\mathcal{C}$. Identify the morphisms which are left adjoints in **Poset**, and in the category **Rel** of sets and relations (with its 'obvious' enrichment over **Poset**).

**6** Explain what is meant by an *exact sequence* and by a *complex* in an abelian category. State the Five Lemma and the Snake Lemma, and use the latter to deduce the Mayer–Vietoris Theorem on the homology objects of an exact sequence of complexes. Hence show that if we are given a commutative diagram of complexes

$$\begin{tikzcd}
0 \arrow[r] & A_\bullet \arrow[r] \arrow[d] & B_\bullet \arrow[r] \arrow[d] & C_\bullet \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & D_\bullet \arrow[r] & E_\bullet \arrow[r] & F_\bullet \arrow[r] & 0
\end{tikzcd}$$

with exact rows, in which any two of the vertical morphisms induce isomorphisms of homology in all dimensions, then the third vertical morphism also induces isomorphisms of homology.

## END OF PAPER