MAMA/116, NST3AS/116, MAAS/116

MAT3
MATHEMATICAL TRIPOS Part III

Monday 10 June 2024 9:00 am to 11:00 am

PAPER 116

LARGE CARDINALS

Before you begin please read these instructions carefully

Candidates have TWO HOURS to complete the written examination.

Attempt **BOTH** questions.
There are **TWO** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 (a) Define the following terms including the definitions of any relevant terms you use in the definition:

(i) strongly inaccessible cardinal,

(ii) weakly compact cardinal,

(iii) measurable cardinal and

(iv) strongly compact cardinal.

[You may assume that the definitions of "κ^+", "2^κ", "$[X]^2$", "singular cardinal", "regular cardinal", "limit cardinal", "filter", "ultrafilter", as well as the syntax and semantics of infinitary languages are known.]

(b) Prove that every measurable cardinal is strongly inaccessible.

[You may assume without proof that measurable cardinals are regular.]

(c) Prove that if κ is strongly compact, then every κ-complete filter on κ can be extended to a κ-complete ultrafilter.

(d) Prove that every strongly compact cardinal is measurable.

[You may assume without proof that strongly compact cardinals are regular.]

2 (a) Explain what the phrase "a β-*strong embedding* reflects β-*stable properties* of κ to an unbounded subset of κ" means, including the definitions of the italicised terms occurring in the phrase.

(b) Let κ be measurable, $\lambda > \kappa$ be inaccessible, and $j\colon \mathbf{V}_\lambda \to M$ be the ultrapower embedding. Furthermore, let η be such that $M \models j(\kappa)^+ = \eta$. Prove the following statements:

(i) If $\kappa < \alpha < \kappa^+$, then $M \models$"α is not a cardinal".

(ii) $\mathbf{V}_\lambda \models$"$\eta$ is not a cardinal".

[You may use results from the lectures without proof provided that you state them correctly.]

(c) Let α be any ordinal, M be a transitive set, and $j\colon \mathbf{V}_\alpha \to M$ be a non-trivial elementary embedding with $\mathrm{crit}(j) = \kappa$. The proof of *Kunen's Inconsistency* gives ordinals β and γ and a set X such that "if $\alpha \geqslant \beta$, then $X \in \mathbf{V}_\gamma \backslash M$". Provide β, γ, and X.

[You do not need to prove the statement.]

(d) Suppose that δ is a limit ordinal and $j\colon \mathbf{V}_\delta \to \mathbf{V}_\delta$ is a non-trivial elementary embedding with $\mathrm{crit}(j) = \kappa < \delta$. Show that $\mathrm{cf}(\delta) = \aleph_0$.

[You may use results from the lectures without proof provided that you state them correctly.]

END OF PAPER