MAMA/115, NST3AS/115, MAAS/115

MAT3

MATHEMATICAL TRIPOS Part III

Friday 7 June 2024 9:00 am to 12:00 pm

PAPER 115

DIFFERENTIAL GEOMETRY

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

Attempt no more than **THREE** questions.
There are **FOUR** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Let M be a manifold. Give three different formulations of *orientability* of M. Prove that the three formulations are equivalent and explain carefully what is meant by an *orientation* of M in each case.

[*Existence of a partition of unity on a manifold can be assumed, provided that you accurately state this result.*]

Prove that if the product $M \times \mathbb{R}$ is orientable, then M is orientable.

Prove that if the product of two manifolds $M \times N$ is orientable, then each of M and N must be orientable.

2 Let M be a manifold. Define the notions *immersed submanifold* of M and *embedded submanifold* of M.

Let $N \subset M$ be an embedded submanifold, $m = \dim M$, $n = \dim N$, and let $p \in N$. Show that there exists a local coordinate chart (ψ, V) on M around p with coordinates $x = (x_i)$ such that $\psi(V \cap N) = \{x \in \psi(V) : x_{n+1} = \ldots = x_m = 0\}$ and $\psi|_{V \cap N}$ is a valid local coordinate chart around p (with coordinates $x_1, \ldots, x_n$) in the differentiable structure on N.

Suppose now that a submanifold $N \subset M$ is only known to be immersed. Suppose further that for each $g \in C^\infty(N)$ there is a C^∞ function f on M such that $f|_N = g$. By considering g supported on a small neighbourhood of a suitable $p \in N$, or otherwise, show that N is an embedded submanifold of M.

[*You may assume that for every $p \in N$ one can choose a coordinate chart (U_p, ϕ_p) on N with $p \in U_p$ so that U_p is an embedded submanifold of M.*]

Let again $N \subset M$ be an embedded submanifold and let X be a vector field on M. We say that X is *tangent to* N if for all $p \in N$ the vector $X(p)$ is in the image of the derivative $(d\iota)_p$ of the inclusion map $\iota : N \to M$. Prove that X is tangent to N if and only if $(Xf)|_N = 0$ for all $f \in C^\infty(M)$ such that $f|_N = 0$.

Define the *Lie bracket* $[X, Y]$ of two vector fields X, Y on M. Give the expression for $[X, Y]$ in local coordinates.

Show that if two vector fields X and Y are tangent to an embedded submanifold $N \subset M$, then $[X, Y]$ is also tangent to N and $[X, Y]|_N$ is determined by $X|_N$ and $Y|_N$.

3 Let $\pi : E \to B$ be a real vector bundle over a manifold B. Define the terms a *horizontal subspace* at $p \in E$ and a *connection* on E.

Let v be a horizontal vector at $p \in E$ (with respect to a given connection on E). By considering lifts of appropriate smooth curves on B, or otherwise, show that there exists a vector field V on E with $V(p) = v$ and such that $V(q)$ is horizontal for every $q \in E$.

Let M be a manifold and $f : M \to B$ a smooth map. By constructing an appropriate family of local trivialisations and transition functions, show that $f^*E = \sqcup_{p \in M} \pi^{-1}(f(p))$ can be made into a manifold with the projection map $\tilde{\pi}(v_p) = p$ for $v_p \in E_{f(p)}$ defining a vector bundle over M, called the *pull-back* of E via f. Given a local section s of E over an open set $U \subset B$, explain how $s \circ f$ uniquely determines a local section of f^*E.

What is a *covariant derivative* on E? Explain briefly how a connection A on E induces a covariant derivative d_A. Suppose that $d_{A'}$ is a covariant derivative induced by a connection on f^*E and

$$\langle d_{A'}(s \circ f), X \rangle = \langle d_A s, (df)X \rangle \circ f$$

holds for all vector fields X on M and sections s of E over B. Choosing appropriate local trivializations of E and of f^*E, determine the coefficients of A' in terms of those of A.

[*You may assume that every covariant derivative is a* local operator *if you define what this means.*]

4 (i) Let (M, g) be an n-dimensional Riemannian manifold. State the ordinary differential equations satisfied by *geodesics* on M in local coordinates, defining clearly all terms appearing in the equation.

Define the *exponential map* at $p \in M$ and prove that the exponential map induces well-defined local coordinates, the *geodesic coordinates*, on some neighbourhood of p.

Now let (φ, U) be a coordinate chart with $\varphi(U)$ a star domain at 0 (for each $x \in \varphi(U)$ the straight line segment from 0 to x lies in $\varphi(U)$) and let $p \in U$, $\varphi(p) = 0$. Prove that φ defines the geodesic coordinates on U if and only if $g_{ij}(0) = \delta_{ij}$ is the identity matrix and

$$\sum_{i,j=1}^{n} x^i x^j \Gamma_{ij}^k(x) = 0$$

for all $k = 1, \ldots, n$ and all $x = (x^i) \in \varphi(U)$.

(ii) Let (M, g) be an oriented Riemannian manifold. Explain how g extends to give an inner product on the bundles of differential forms on X. Define the *Hodge $*$-operator* for the differential forms on X and compute its square on r-forms. (You do not need to define the volume form of g.)

Now assume in addition that M is compact and $\dim M = 4$. Let η be a differential 2-form on X. Prove, without using the Hodge decomposition theorem, that if $*d\delta\eta = -d\delta\eta$, then $\delta\eta = 0$.

[*You may assume that $\delta = -*d*$ defines the formal L^2 adjoint of the exterior derivative d in 4 dimensions.*]

END OF PAPER