UNIVERSITY OF CAMBRIDGE

MAMA/114

MAT3

MATHEMATICAL TRIPOS Part III

Thursday 30 May 2024 9:00 am to 12:00 pm

PAPER 114

ALGEBRAIC TOPOLOGY

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

Attempt **ALL** questions.
There are **FOUR** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Let $n > 1$ be a positive integer. Let Y be the quotient space of a closed two-dimensional disc D^2 by identifying points on the boundary ∂D^2 by the rotation action $z \mapsto e^{2i\pi/n} \cdot z$. Compute $H_*(Y;\mathbb{Z})$.

Suppose that the continuous map $f : Y \to Y$ induces a surjection on $H_*(Y;\mathbb{Z})$. Prove that f is itself a surjection. Is the converse true?

Let $Z \subset Y$ be the image of ∂D^2 in Y. When $n > 2$, show that any homeomorphism $f : Y \to Y$ must satisfy $f(Z) \subset Z$. Does the same conclusion hold when $n = 2$? Briefly justify your answer.

2 Let $f : X \to Y$ be a double covering of topological spaces, so f is a local homeomorphism and $f^{-1}(y)$ consists of two points for every $y \in Y$. Construct a short exact sequence of chain complexes

$$0 \to C_*(Y;\mathbb{Z}/2) \to C_*(X;\mathbb{Z}/2) \to C_*(Y;\mathbb{Z}/2) \to 0$$

explaining carefully the maps which arise.

[*You may assume that if D is homeomorphic to a closed disc (of any dimension) and $\sigma : D \to Y$ is continuous, then for $x \in D$ and $p \in f^{-1}(\sigma(x))$, there is a unique continuous map $\tilde{\sigma} : D \to X$ with $\tilde{\sigma}(x) = p$ and $f \circ \tilde{\sigma} = \sigma$.*]

By considering the corresponding long exact sequence in homology, or otherwise, prove that if $\phi : \mathbb{R}^n \to \mathbb{R}^n$ is a continuous involution, i.e. satisfies $\phi \circ \phi = \mathrm{id}$, then ϕ has a fixed point.

Let M be a compact manifold of odd dimension $2k+1$, and remove an open ball from M to give a manifold P with boundary $\partial P \cong S^{2k}$. By considering Euler characteristics, or otherwise, show that the antipodal map on S^{2k} does not extend to a continuous fixed-point-free involution of P.

3 Let $0 < k < n$. Compute the integral cohomology of the quotient space $\mathbb{CP}^n/\mathbb{CP}^k$ and determine the ring structure in cohomology. (You do not need to give a closed formula for the cohomology ring.) For which n and k is this space homotopy equivalent to a compact manifold?

Let ι denote the inclusion map $\iota : \mathbb{CP}^2/\mathbb{CP}^1 \hookrightarrow \mathbb{CP}^4/\mathbb{CP}^1$. Show there is no continuous map $r : \mathbb{CP}^4/\mathbb{CP}^1 \to \mathbb{CP}^2/\mathbb{CP}^1$ for which $r \circ \iota \simeq \mathrm{id}$.

4 Compute $H^*(S^n \times S^n; \mathbb{Z})$ as a ring. Explain how the naturality of cohomology defines a homomorphism from the group of homeomorphisms of $S^n \times S^n$ to $GL(2; \mathbb{Z})$. Find the image of this homomorphism when $n = 1$ and when $n = 2$.

If L is a complex line bundle with dual bundle L^*, prove that $L^* \otimes_{\mathbb{C}} L$ is a trivial line bundle. Hence, or otherwise, show that every class $\alpha \in H^2(S^2 \times S^2; \mathbb{Z})$ arises as the Euler class of an oriented real rank two vector bundle $E_\alpha \to S^2 \times S^2$. [*You may assume the formula* $e(L_1 \otimes_{\mathbb{C}} L_2) = e(L_1) + e(L_2)$ *for Euler classes of complex line bundles.*]

Writing $S(E_\alpha)$ for the sphere bundle of E_α, compute $H^*(S(E_\alpha); \mathbb{Z})$ additively. Deduce that this cohomology does not determine the orbit of α under the homeomorphism group.

END OF PAPER