UNIVERSITY OF CAMBRIDGE

MAMA/113, NST3AS/113, MAAS/113

MAT3

MATHEMATICAL TRIPOS Part III

Monday 10 June 2024 1:30 pm to 4:30 pm

PAPER 113

ALGEBRAIC GEOMETRY

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

Attempt **ALL** questions.
There are **THREE** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Throughout this question the symbol k will be used to denote a field and the term "coherent sheaf" will mean "coherent sheaf of $\mathcal{O}_X$-modules".

(a) Define *separatedness* for a morphism of schemes. Given an example of a scheme X and two affine opens $U \subset X$ and $V \subset X$ such that $U \cap V$ is not affine. Comment on whether the chosen example X is separated.

(b) Equip the polynomial ring $A_\bullet = k[x, y]$ with the standard grading, and define $\mathbb{P}^1_k$ to be $\mathsf{Proj}\ A_\bullet$. Let $M = A_\bullet(d)$ be the graded $A_\bullet$-module whose degree m part M_m is the degree $m + d$ part A_{m+d} of $A_\bullet$. Let $\mathcal{O}(d)$ be the coherent sheaf associated to $A_\bullet(d)$. By describing the sections of this sheaf, calculate the dimension of the vector space $H^0(\mathbb{P}^1_k, \mathcal{O}(d))$ for all d.

Let X be a scheme and let $\{U_i\}_{i \in I}$ be an open affine cover. Let $\mathcal{F}$ and $\mathcal{G}$ be coherent sheaves such that, for all $i \in I$, the restrictions $\mathcal{F}|_{U_i}$ and $\mathcal{G}|_{U_i}$ are isomorphic. Is it necessarily true that $\mathcal{F}$ is isomorphic to $\mathcal{G}$?

Let $\mathcal{H}$ and $\mathcal{K}$ be line bundles (i.e. locally free sheaves of rank 1) on a scheme Y and let $\varphi \colon \mathcal{H} \to \mathcal{K}$ be an injective morphism of coherent sheaves. Is φ necessarily an isomorphism?

(c) Let p be a closed k-point in $\mathbb{P}^n_k$ for $n \geqslant 2$, let $X = \mathbb{P}^n_k \backslash \{p\}$, and let $\pi \colon X \to \mathrm{Spec}(k)$ be the natural morphism. Calculate the pushforward $\pi_\star \mathcal{O}_X$ and comment on whether the result is a coherent sheaf on $\mathrm{Spec}(k)$.

Let Y be the complement of a closed k-point in $\mathbb{P}^1_k$. Is the pushforward of $\mathcal{O}_Y$ under $Y \to \mathrm{Spec}(k)$ coherent?

Give an example of a coherent sheaf $\mathcal{F}$ on X such that $\pi_\star \mathcal{F}$ is not coherent.

2

(a) Define the fibre product of schemes and state what it means for a morphism to be *universally closed.*

Give an example of a morphism that is closed but not universally closed.

Give an example of a morphism of schemes $X \to Y$ that is of finite type and universally closed, but not separated. Justify your answer.

(b) Let $X \to S$ be a morphism of schemes. Prove that diagonal morphism $X \to X \times_S X$ is always a locally closed immersion.

Let k be a field and let $f \colon \mathbb{A}^1_k \to \mathsf{Spec}(k)$ be the morphism induced by the inclusion $k \hookrightarrow k[x]$. Let Δ denote the image of the diagonal morphism

$$\mathbb{A}^1_k \to \mathbb{A}^1_k \times_{\mathsf{Spec}(k)} \mathbb{A}^1_k.$$

Let U denote the complement of Δ with the induced open subscheme structure. Show that U is isomorphic to an affine scheme.

Give an example of a separated scheme X over $\mathsf{Spec}(k)$ such that the complement of the diagonal in $X \times_{\mathsf{Spec}(k)} X$ is not affine.

(c) What is the Weil divisor class group of a (noetherian, separated, integral, regular in codimension 1) scheme?

Suppose X is a scheme with trivial class group and let $U \subset X$ be an open subscheme. Then does U also have trivial class group? Justify your answer.

Prove or give a counterexample: all affine schemes have trivial class group.

3

Throughout this question, the symbol k will be used to denote a field.

(a) What is a quasi-coherent $\mathcal{O}_X$-module on a scheme X?

Let $X = Y = \mathbb{P}^1_k$. Consider the morphism $f\colon X \to Y$ given in homogeneous coordinates by

$$[x_0, x_1] \mapsto [x_0^2, x_1^2].$$

Prove that $f_\star \mathcal{O}_X$ is a locally free $\mathcal{O}_Y$-module of rank 2.

(b) Suppose W is the closed subscheme in $\mathbb{A}^n_k$ defined by $\mathbb{V}(f_1, \ldots, f_m)$. Let Y denote the complement of W. Prove that the Cech cohomology group $H^m(Y, \mathcal{O}_Y)$ vanishes.

Let x, y, z denote the standard coordinate functions on $\mathbb{A}^3_k$. Let $p \in \mathbb{A}^3_k$ be a closed k-point and let $\ell \subset \mathbb{A}^3_k$ be the closed subscheme defined by the ideal (x, y). By using Cech cohomology or otherwise, prove that the schemes $\mathbb{A}^3_k \setminus \{p\}$ and $\mathbb{A}^3_k \setminus \ell$ are not isomorphic.

(c) Suppose k is algebraically closed. Say a scheme Z over $\mathsf{Spec}(k)$ is *punctual* if its underlying topological space is a single closed k-point. Give, with justification, two non-isomorphic punctual schemes over $\mathsf{Spec}(k)$.

Let Z and Z' be closed subschemes of $\mathbb{A}^1_k$ that are both punctual, with the same underlying topological space. Suppose $H^0(Z, \mathcal{O}_Z)$ and $H^0(Z', \mathcal{O}_{Z'})$ have the same dimension as vector spaces over k. Prove that Z and Z' are isomorphic as closed subschemes of $\mathbb{A}^1_k$.

Give an example of two distinct punctual closed subschemes Y and Y' of $\mathbb{A}^2_k$ with the same underlying topological space, and such that $H^0(Y, \mathcal{O}_Y)$ and $H^0(Y', \mathcal{O}_{Y'})$ have the same dimension as vector spaces over k.

END OF PAPER