UNIVERSITY OF CAMBRIDGE

MAMA/109, NST3AS/109, MAAS/109

MAT3
MATHEMATICAL TRIPOS Part III

Monday 10 June 2024 1:30 pm to 3:30 pm

PAPER 109

COMBINATORICS

Before you begin please read these instructions carefully

Candidates have TWO HOURS to complete the written examination.

Attempt no more than **THREE** questions.
There are **FOUR** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(a) Show that for $n \geqslant 1$, the power set $\mathcal{P}(n)$ has a symmetric chain decomposition.

(b) Let $X_1 \cup X_2$ be a partition of a set X with $|X| = n$ and $|X_i| = n_i$ for $i = 1, 2$, with n_1 and n_2 even. Let $\mathcal{F} \subset \mathcal{P}(X)$ be such that if $E, F \in \mathcal{F}$ with $E \subsetneq F$ then $F \backslash E \not\subset X_i$, $i = 1, 2$. Show that $|\mathcal{F}| \leqslant \binom{n}{n/2}$.

2

Let p be a prime and consider the vector space $V = \mathbb{Z}_p \oplus \mathbb{Z}_p$ over $\mathbb{Z}_p$.

(a) Let $(v_i)_1^m$, $v_i = (a_i, b_i)$, be a sequence of length $m = 3p$ whose members are vectors in V summing to 0. Show that some p members of this sequence also sum to 0, i.e. there is a p-subset $I \subset [m]$ such that $\sum_{i \in I} v_i = 0$.

(b) Now let $(v_i)_1^m$, $v_i = (a_i, b_i)$, be a sequence of length $m = 4p - 2$ whose members are vectors in V. Show that some p members of this sequence sum to 0.

[*If you wish to use a theorem from algebra, you must state it precisely.*]

3

(a) For non-negative integers $a_1, \ldots, a_n$, let $D(a) = D(a_1, \ldots, a_n)$ be the constant term in the expansion of the n-variable Laurent polynomial

$$\prod_{i=1}^{n} \prod_{\substack{1 \leqslant j \leqslant n \\ j \neq i}} \left(1 - \frac{X_i}{X_j}\right)^{a_i}.$$

Show that $D(a)$ is equal to the multinomial coefficient

$$M(a) = \binom{m}{a_1, \ldots, a_n},$$

where $m = a_1 + \cdots + a_n$.

(b) Let G be the group $(\mathbb{Z}_p)^\gamma$, where $p \geqslant 3$ is a prime and $\gamma \geqslant 1$ is an integer. Let $A = \{a_1, \ldots, a_k\}$ and B be subsets of G, each with $k < p$ elements. Show that there is a numbering $b_1, \ldots, b_k$ of the elements of B, such that the sums $a_1 + b_1, \ldots, a_k + b_k$ are all different.

[*Hint: Identify G with the additive group of the finite field $\mathbb{F}_q$ of order $q = p^\gamma$.*]

4

(a) Show the equivalence of the following three assertions.

(i) Every antipodal map $f : S^n \to \mathbb{R}^n$ maps a point into 0.

[Here as usual we say that f is 'antipodal' if it is continuous with $f(-x) = -f(x)$ for every $x \in S^n$.]

(ii) There is no antipodal map from S^n to S^{n-1}.

(iii) There is no continuous map from B^n to S^{n-1} which is antipodal on the boundary of B^n.

(b) Let $\mathcal{C}_1, \ldots, \mathcal{C}_n$ be families of intersecting convex compact sets in $\mathbb{R}^n$. (Thus $A \cap B \neq \emptyset$ if $A, B \in \mathcal{C}_i$ for some i.) Show that there is a hyperplane in $\mathbb{R}^n$ which intersects every set in $\cup_{i=1}^n \mathcal{C}_i$.

[In your answer to Part (b), you may use without proof that the statements in Part (a) are true.]

[*Hint: Given a unit vector $v \in \mathbb{R}^n$, let ℓ_v be the line in the direction of v through the origin. The projections of the sets in $\mathcal{C}_i$ into ℓ_v intersect in an interval: write $m_i(v)$ for the midpoint of this interval, and $f_i(v)$ for the signed distance of $m_i(v)$ from the origin. Show that there is a unit vector $v \in S^{n-1}$ such that $f_1(v) = \cdots = f_n(v)$, and make use of this.*]

END OF PAPER