UNIVERSITY OF CAMBRIDGE

MAMA/107, NST3AS/107, MAAS/107

MAT3

MATHEMATICAL TRIPOS Part III

Monday 10 June 2024 9:00 am to 12:00 pm

PAPER 107

ELLIPTIC PARTIAL DIFFERENTIAL EQUATIONS

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

Attempt no more than **THREE** questions.
There are **FOUR** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Let $\alpha \in (0,1)$, $0 < \beta < \infty$, and

$$L = a^{ij}(x)\partial^2_{ij} + b^i(x)\partial_i + c(x)$$

be a strictly elliptic differential operator on $B_1(0) \subset \mathbb{R}^n$ such that

$$|a^{ij}|_{0,\alpha;B_1(0)} + |b^i|_{0,\alpha;B_1(0)} + |c|_{0,\alpha;B_1(0)} \leqslant \beta$$

and $a^{ij}(x) = a^{ji}(x)$ for all $i, j \in \{1,\ldots,n\}$. Suppose $u \in C^{2,\alpha}(B_1(0)) \cap C^0(\overline{B_1(0)})$ satisfies $Lu = f \in C^{0,\alpha}(B_1(0))$ in $B_1(0)$.

(a) State what it means for L to be strictly elliptic. State what it means for L to be uniformly elliptic and explain why here L is in fact uniformly elliptic.

(b) State, without proof, the interpolation inequality in Hölder spaces for a function $u \in C^{l,\alpha}(\overline{B_R(x_0)})$, $l \in \mathbb{N}$.

(c) State, without proof, Simon's Absorbing Lemma for a non-negative function S on a ball $B_R(x) \subset \mathbb{R}^n$ which is sub-additive on sub-balls of $B_R(x)$.

(d) State and prove the $C^{2,\alpha}$ interior Schauder estimate in $B_1(0)$ for u.

(e) Comment briefly on which steps in your proof fail if $\alpha \in \{0,1\}$.

[*Hint: You may use the Arzelà–Ascoli Theorem in Hölder spaces without proof. You may also use without proof the fact that $C^{2,\alpha}(\mathbb{R}^n)$ harmonic functions are smooth, as well as Liouville's Theorem for harmonic functions. In part (d), recall that the proof proceeds in three steps: reduction, contradiction, and a consideration of a function obtained in the limit $k \to \infty$ for a suitable index k. In step three you may assume without proof that*

$$g_k = \frac{\tilde{f}_k - \tilde{L}_k q_k}{\rho_k^{2+\alpha}[D^2 u_k]_{\alpha;B_1}}$$

tends locally uniformly to zero, where $\tilde{f}_k$, $\tilde{L}_k$ are sequences of appropriately localised functions and operators, q_k is the Taylor expansion up to second order of u_k around an appropriate point, and u_k and ρ_k are functions you should define in your proof. Additionally, you may use without justification the fact that the strictly elliptic constant coefficient second order PDE $\tilde{a}^{ij}\partial^2_{ij}v = 0$ may be written as $\Delta\tilde{w} = 0$, where $\tilde{w}$ is related to v by a rotation and scaling of the basis vectors.]

2 Let $\Omega \subset \mathbb{R}^n$, be an open and bounded domain with smooth boundary. Consider a pair of functions $(X, Y) : \Omega \to \mathbb{H}$, where $\mathbb{H} = \{(\xi, \eta) \in \mathbb{R}^2 \, : \, \xi > 0\}$ is the open right half-plane, and consider the problem of minimizing the functional

$$E[X, Y] = \int_\Omega \frac{|\nabla X|^2 + |\nabla Y|^2}{X^2} \, \mathrm{d}x,$$

assuming throughout the question that there exists a constant $C > 0$ such that

$$C^{-1} \leqslant X \leqslant C$$

in Ω, i.e. that $X \in L^\infty(\Omega)$ and $X^{-1} \in L^\infty(\Omega)$.

(a) Use the Euler–Lagrange equations to show that smooth extremizers of $E[X, Y]$ obey

$$\Delta X = \frac{|\nabla X|^2 - |\nabla Y|^2}{X}, \tag{1}$$

$$\operatorname{div}\left(\frac{\nabla Y}{X^2}\right) = 0 \tag{2}$$

in Ω.

(b) By expanding $E[X + t\varphi, Y + t\psi]$ in t for $(X, Y) \in H^1(\Omega; \mathbb{H})$ and arbitrary $(\varphi, \psi) \in C_c^\infty(\Omega; \mathbb{H})$, obtain a weak formulation of (1)–(2). Confirm that when $(X, Y) \in C^\infty(\Omega; \mathbb{H})$, this coincides with the equations (1)–(2).

(c) Given $(X_0, Y_0) \in C^\infty(\overline{\Omega}; \mathbb{H})$, let

$$W = \{(X, Y) \in H^1(\Omega; \mathbb{H}) \, : \, (X - X_0, Y - Y_0) \in H_0^1(\Omega; \mathbb{H})\}.$$

Use the Direct Method of Calculus of Variations to show that there exists a weak solution $(X, Y) \in W$ to the system (1)–(2). [*You may use without proof the facts that* $H_0^1(\Omega; \mathbb{H})$ *is weakly closed and that* $E[X, Y]$ *is sequentially weakly lower semicontinuous with respect to convergence in* $H^1(\Omega; \mathbb{H})$.]

The higher interior regularity Schauder estimates for strictly elliptic operators in non-divergence form on Ω state that if a^{ij}, b^i, $c \in C^{k,\alpha}(\Omega)$ for some $k \geqslant 0$ and $\alpha \in (0, 1)$, and the operator $L = a^{ij}\partial_{ij}^2 + b^i\partial_i + c$ is strictly elliptic in Ω and $f \in C^{k,\alpha}(\Omega)$, then if u solves $Lu = f$, then

$$|u|_{k+2,\alpha;\Omega'} \leqslant C\left(|u|_{0;\Omega_1} + |f|_{k,\alpha;\Omega_1}\right)$$

for any $\Omega' \subset\subset \Omega_1 \subset\subset \Omega$ and some constant $C > 0$ independent of f or u.

(d) State, without proof, the interior $C^{1,\alpha}$ Schauder estimate for a weak solution of a strictly elliptic equation in *divergence* form on Ω.

(e) You are given that weak solutions to (1)–(2) are $C^{0,\alpha}(\Omega')$ for any $\Omega' \subset\subset \Omega$ and some $\alpha \in (0, 1)$. By rewriting (1) as an equation for $\log X$, explain why the solution constructed in (b) is in fact $C^\infty(\Omega'; \mathbb{H})$ for $\Omega' \subset\subset \Omega$.

3 Throughout this question, $n \geqslant 2$ and B_ρ denotes the open ball in $\mathbb{R}^n$ with radius ρ and centre the origin.

(a) Let $a^{ij} \in L^\infty(B_1)$ for $1 \leqslant i,j \leqslant n$, and suppose that there are constants $\lambda, \Lambda > 0$ such that $a^{ij}(x)\zeta^i\zeta^j \geqslant \lambda|\zeta|^2$ for a.e. $x \in B_1$ and all $\zeta \in \mathbb{R}^n$, and $\sum_{i,j=1}^n \|a^{ij}\|_{L^\infty(B_1)} \leqslant \Lambda^2$. State without proof the interior *De Giorgi–Nash–Moser theorem* concerning continuity of a weak solution $u \in W^{1,2}(B_1)$ to $D_i\left(a^{ij}D_j u\right) = 0$ in B_1, giving the relevant estimate describing continuity of u in B_θ for any fixed $\theta \in (0,1)$.

[*In your estimate, you need not provide explicit dependence of the constants on the given parameters, but you should specify which parameters the constants depend on.*]

(b) Let $\mathcal{A}(v) = \int_{B_1} \sqrt{1+|Dv|^2}$ be the area functional associated with functions $v \in C^1(\overline{B_1})$. Suppose that $u \in C^2(B_1) \cap C^1(\overline{B_1})$ is a minimiser of $\mathcal{A}(\cdot)$ in the sense that $\mathcal{A}(u) \leqslant \mathcal{A}(v)$ for any $v \in C^1(\overline{B_1})$ with $u = v$ on ∂B_1.

(i) Show that u satisfies the *minimal surface equation*

$$\left(\delta_{ij} - \frac{D_i u D_j u}{1+|Du|^2}\right) D_{ij}u = 0 \quad \text{in } B_1.$$

Show further that for each $k \in \{1,2,\ldots,n\}$, the partial derivative $w = D_k u$ satisfies an equation of the form $D_i\left(a^{ij}(Du)D_j w\right) = 0$ in B_1, where $a^{ij} : \mathbb{R}^n \to \mathbb{R}$. Give an explicit expression for $a^{ij}(p)$ in terms of $p = (p_1, p_2, \ldots, p_n) \in \mathbb{R}^n$.

(ii) Let $L > 0$ be a constant and suppose that $\sup_{B_1} |Du| \leqslant L$. Let $b^{ij}(x) = \delta_{ij} - \frac{D_i u(x) D_j u(x)}{1+|Du(x)|^2}$. Show that there is a constant $\alpha = \alpha(n,L) \in (0,1)$ such that for each $i,j \in \{1,2,\ldots,n\}$, $|b^{ij}|_{0,\alpha;B_\theta} \leqslant \beta$ for each $\theta \in (0,1)$ and some constant $\beta = \beta(n,L,\theta) \in (0,\infty)$.

(iii) By considering the Dirichlet problem for the linear equation $b^{ij}D_{ij}v = 0$ in an appropriate ball with appropriate boundary data, or otherwise, show that $u \in C^{2,\alpha}(B_1)$. Show further that $|u|_{2,\alpha;B_\theta} \leqslant C\|u\|_{L^2(B_1)}$ for a constant $C = C(n,L,\theta)$.

[*You may use, without proof but with clear statements, standard existence and regularity theorems for solutions to PDEs proved in the course.*]

(c) Let $(u_k)_{k=1}^\infty$ be a sequence of non-zero functions in $C^2(B_1) \cap C^1(\overline{B_1})$ satisfying the minimal surface equation in B_1. If $\sup_k |Du_k|_{0;B_1} < \infty$ and $\|u_k\|_{L^2(B_1)} \to 0$, show that there is a C^2 harmonic function w on B_1 and a subsequence $(u_{k'})$ such that

$$\frac{u_{k'}}{\|u_{k'}\|_{L^2(B_1)}} \to w$$

in $C^2(K)$ for every compact set $K \subset B_1$.

4 Throughout this question, $n \geqslant 2$ and B_ρ denotes the open ball in $\mathbb{R}^n$ with radius ρ and centre the origin. Let $q : B_2 \to \mathbb{R}$ be a bounded function with $\sup_{B_2} |q| \leqslant \mu$ for some constant μ, and let $u : B_2 \to \mathbb{R}$ be a non-negative C^2 function satisfying

$$\Delta u + qu \leqslant 0 \text{ in } B_2.$$

Let $u_\epsilon = u + \epsilon$ for constant $\epsilon > 0$.

(a) By considering $w = \log u_\epsilon$ or otherwise, establish the following:

(i) for any $\zeta \in C_c^1(B_2)$,

$$\int_{\{x \in B_2 \,:\, u(x) > 0\}} q\zeta^2 \leqslant \int |D\zeta|^2.$$

(ii) for any ball $B_\rho(z)$ and some fixed constant $K = K(n, \mu) \in (0, \infty)$,

$$\rho^{2-n} \int_{B_\rho(z) \cap B_1} \frac{|Du_\epsilon|^2}{u_\epsilon^2} \leqslant K.$$

(b) The John–Nirenberg lemma says the following: *there are constants* $p = p(n) > 0$ *and* $C = C(n) > 0$ *such that if* $v \in W^{1,1}(B_1)$ *satisfies* $\rho^{1-n} \int_{B_\rho(z) \cap B_1} |Dv| \leqslant M$ *for some constant* M *and all balls* $B_\rho(z)$*, then* $\int_{B_1} e^{\frac{p}{M}|v - v_a|} \leqslant C$*, where* $v_a = \frac{1}{|B_1|} \int_{B_1} v$. Deduce the following from the results of (a) and the John–Nirenberg lemma (without appealing to any other results from the course unless you prove them).

(i) There are constants $p_0 = p_0(n, \mu) > 0$ and $C = C(n)$ such that

$$\left(\int_{B_1} u_\epsilon^{-p_0} \right) \left(\int_{B_1} u_\epsilon^{p_0} \right) \leqslant C.$$

(ii) If $u \equiv 0$ on a set $\Sigma \subset B_1$ with $|\Sigma| > 0$, then $u \equiv 0$ on B_1. Here $|\Sigma|$ denotes the Lebesgue measure of Σ.

(iii) If $\inf_{B_2} q > \lambda_1$, where λ_1 is the first Dirichlet eigenvalue of B_2 characterised by

$$\lambda_1 = \inf\{ \int_{B_2} |D\zeta|^2 \,:\, \zeta \in C_c^1(B_2), \;\; \|\zeta\|_{L^2(B_2)} = 1 \},$$

then any function $w \in C^2(B_2)$ solving $\Delta w + qw = 0$ in B_2 which is not identically zero must take both positive and negative values in B_2.

(c) Must the conclusion in (b)(ii) hold under the weaker hypothesis that $u(y) = 0$ for some $y \in B_1$? Give a brief explanation for your answer, quoting without proof any necessary results from the course.

END OF PAPER