UNIVERSITY OF CAMBRIDGE

MAMA/106, NST3AS/106, MAAS/106

MAT3

MATHEMATICAL TRIPOS Part III

Monday 3 June 2024 1:30 pm to 4:30 pm

PAPER 106

FUNCTIONAL ANALYSIS

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

Attempt **ALL** questions.
There are **FOUR** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Throughout this question, any results from the course may be used without proof provided they are clearly stated.

(a) Let K be a weakly compact subset of a Banach space X. Prove that K is weakly sequentially compact. [Hint: First show that under the additional assumption that X is separable, K is weakly metrizable.]

(b) Let X and Y be Banach spaces and $T\colon X \to Y$ be a compact operator. Assume that X is reflexive. Show that for any bounded sequence (x_n) in X, there is a subsequence (x_{k_n}) of (x_n) and an element $z \in X$ such that $Tx_{k_n} \to Tz$ as $n \to \infty$.

(c) A Banach space X is called a *Grothendieck space* if every w^*-convergent sequence in X^* is weakly convergent.

Show that every reflexive space is a Grothendieck space. Conversely, show that if X is a separable Grothendieck space, then X is reflexive. [You may assume the converse of part (a): a weakly sequentially compact subset of a Banach space is weakly compact.]

Show that if $T\colon X \to Y$ is a bounded linear map from a Grothendieck space X onto a Banach space Y, then Y is also a Grothendieck space.

2

Throughout this question, A is a unital C^*-algebra. You may assume any result about general Banach algebras but no results specific to C^*-algebras.

Define what it means for an element $x \in A$ to be, respectively, *hermitian, unitary, normal* and *positive*.

Let $\tau \in A^*$ be a linear functional with $\|\tau\| = \tau(\mathbf{1})$. Show that $\tau(x) \in \mathbb{R}$ for every hermitian $x \in A$. [Hint: Consider $x + \mathrm{i}t\mathbf{1}$ for $t \in \mathbb{R}$.] Deduce that $\tau(x^*) = \overline{\tau(x)}$ for every $x \in A$.

Let $x \in A$ be hermitian. Prove that $\sigma_A(x) \subset \mathbb{R}$ and that $\|x\| = r_A(x)$. Show further that x may be written as a difference of two positive elements of A.

Let $x \in A$ be positive with $\|x\| \leqslant 1$. Show that $\mathbf{1} - x$ is positive and $\|\mathbf{1} - x\| \leqslant 1$.

A bounded linear functional $\tau \in A^*$ is said to be *positive* if $\tau(x)$ is real and non-negative for every positive $x \in A$. Show that if $\tau \in A^*$ satisfies $\|\tau\| = \tau(\mathbf{1})$, then τ is positive. Deduce that every character $\varphi \in \Phi_A$ is positive. Give an example of a positive functional on $C[0,1]$ that is not a character.

Let $\tau \in A^*$ be a positive functional. Show that $\tau(x) \in \mathbb{R}$ for every hermitian $x \in A$. Explain briefly why the inequality

$$|\tau(y^*x)| \leqslant \tau(x^*x)^{1/2}\tau(y^*y)^{1/2}$$

holds for all $x, y \in A$. [You may assume that z^*z is positive for every $z \in A$.] Deduce that $|\tau(x)|^2 \leqslant \tau(x^*x)\tau(\mathbf{1})$ for every $x \in A$ and that $\|\tau\| = \tau(\mathbf{1})$.

A positive functional $\tau \in A^*$ with $\|\tau\| = 1$ is called a *state*. Show that the set $S(A)$ of states of A is convex. Show that for every normal element $x \in A$ there exists a state τ with $|\tau(x)| = \|x\|$. [Hint: Consider the unital C^*-subalgebra of A generated by x and use Hahn–Banach.]

An extreme point of $S(A)$ is called a *pure state*. Show that the set $PS(A)$ of all pure states of A is non-empty. Show that if $x \in A$ is positive, then there is a pure state τ with $\tau(x) = \|x\|$. [Hint: Consider a suitable face of $S(A)$.]

3

State and prove the Hahn–Banach separation theorem for a disjoint pair of non-empty convex sets A and B in a real locally convex space X with A open. [You may assume any form of the Hahn–Banach extension theorems. You must prove any other results used.]

Let E be a real vector space and let F be a subspace of linear functionals on E that separates the points of E: for every non-zero $x \in E$, there exists $f \in F$ with $f(x) \neq 0$. Call such a pair (E, F) a *dual pair*. Define the weak topology $\sigma(E, F)$ on E. Show that $\big(E, \sigma(E, F)\big)^* = F$. Define the *weak topology* of a normed space and the *weak-star topology* of a dual space. Show that a Banach space X is reflexive if and only if the weak and weak-star topologies on X^* coincide.

Let (E, F) be a dual pair as above. For $A \subset E$ define

$$A^\circ = \{f \in F : \, f(x) \leqslant 1 \text{ for all } x \in A\} \ ,$$

and for $B \subset F$ define

$$B^\circ = \{x \in E : \, f(x) \leqslant 1 \text{ for all } f \in B\} \ .$$

Given $A \subset E$, show that $A \cup \{0\} \subset A^{\circ\circ}$ and identify, with proof, the set $A^{\circ\circ}$.

4

Let K be a compact Hausdorff space. Define the set $M^+(K)$ of *positive linear functionals*. Show that if $\varphi \in M^+(K)$, then φ is continuous with $\|\varphi\| = \varphi(\mathbf{1}_K)$. State the Riesz Representation Theorem for positive linear functionals on $C(K)$. Describe, without proof, the dual space $M(K)$ of $C(K)$.

Let H be a non-zero complex Hilbert space. Let A be a commutative, unital C^*-subalgebra of $\mathcal{B}(H)$ and let $K = \Phi_A$. Prove that there is a norm-decreasing, unital $*$-homomorphism $\Psi \colon L_\infty(K) \to \mathcal{B}(H)$ such that $\Psi(\widehat{T}) = T$ for all $T \in A$, where $T \mapsto \widehat{T}$ is the Gelfand map $A \to C(K)$.

Let $T \in \mathcal{B}(H)$ be a normal operator, and let $K = \sigma(T)$. Show that there is a norm-decreasing, unital $*$-homomorphism $\Psi \colon L_\infty(K) \to \mathcal{B}(H)$ such that $\Psi(z) = T$, where $z(\lambda) = \lambda$ for all $\lambda \in K$. Show that if $K \subset \mathbb{T} = \{\lambda \in \mathbb{C} : \, |\lambda| = 1\}$, then T is unitary.

END OF PAPER