UNIVERSITY OF CAMBRIDGE

MAMA/105, NST3AS/105, MAAS/105

MAT3

MATHEMATICAL TRIPOS Part III

Wednesday 5 June 2024 9:00 am to 12:00 pm

PAPER 105

ANALYSIS OF PARTIAL DIFFERENTIAL EQUATIONS

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

Attempt **ALL** questions.
There are **THREE** questions in total.
Question 1 carries 40 marks. Questions 2 and 3 carry 30 marks.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(a) Consider the following semilinear equation for $u : \mathbb{R}^2 \to \mathbb{R}$,

$$tx\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} + \left(\frac{\partial u}{\partial t}\right)^2 = 0\,. \qquad (*)$$

(i) Find the region(s) in $\mathbb{R}^2$ where $(*)$ is hyperbolic.

(ii) Find functions $p = p(t)$ and $q = q(x)$ such that the characteristic surfaces are curves of constant $p \pm q$.

(iii) Consider equation $(*)$ with the data

$$u(t_0, x) = \sin(x)\,, \quad u_t(t_0, x) = 0\,, \quad x \in \mathbb{R}\,. \qquad (**)$$

For which $(t_0, x_0) \in \mathbb{R}^2$ does the Cauchy–Kovalevskaya theorem imply that the above initial value problem, $(*)$ and $(**)$, has an analytic solution in a neighbourhood of (t_0, x_0)?

(b) (i) Define what it means for a function $u \in L^1_{loc}(\mathbb{R})$ to have a *weak derivative.*

(ii) Does the function

$$u : (-\pi, \pi) \to \mathbb{R} : \quad x \mapsto \sin(|x|)$$

have a weak derivative? Justify your answer.

(c) (i) State the Gagliardo-Nirenberg-Sobolev inequality for $W^{1,p}(\mathbb{R}^n)$.

(ii) Let $U \subset \mathbb{R}^n$ be open and bounded with smooth boundary and $1 \leqslant p < n$. Use your answer from part (c.i) to show that there exist positive constants C, C' such that

$$C\|Du\|_{L^p(U)} \leqslant \|u\|_{W^{1,p}(U)} \leqslant C'\|Du\|_{L^p(U)}\,, \quad \forall u \in W_0^{1,p}(U)\,.$$

(d) (i) Consider the BVP

$$\begin{cases} Lu = f & \text{in } U\,, \\ u = 0 & \text{on } \partial U\,, \end{cases} \qquad (\dagger)$$

for L a uniformly elliptic operator and $U \subset \mathbb{R}^n$ an open, bounded set with C^1-boundary. State the Fredholm Alternative for $(\dagger)$.

(ii) Let $U = (0, 2\pi) \subset \mathbb{R}$ and define the operator

$$Lu := \frac{d^2 u}{dx^2} + u\,.$$

Consider the boundary value problem

$$\begin{cases} Lu = f & \text{in } U\,, \\ u = 0 & \text{on } \partial U\,. \end{cases} \qquad (\dagger\dagger)$$

Use your answer from part (d.i) to determine whether a solution to $(\dagger\dagger)$ exists when $f(x) = \cos(x)$. [40]

2 Let $I = (0, 1) \subset \mathbb{R}$ and $1 < p < \infty$.

(a) Given $u \in L^p(I)$ define

$$Au(x) = \frac{1}{x} \int_0^x u(t)\mathrm{d}t\,, \qquad \text{for } x \in (0, 1)\,.$$

Show that $Au \in L^p(I)$ with

$$\|Au\|_{L^p(I)} \leqslant \frac{p}{p-1} \|u\|_{L^p(I)}\,.$$

[Hint: you may wish to consider the function $\phi(x) = \int_0^x u(t)dt$ and estimate $\|Au\|_{L^p}$ in terms of ϕ and its derivatives. Note also the identity $x^{-p}dx = \frac{-1}{p-1} d(x^{1-p})$.]

(b) Let $u \in W^{1,p}(I)$ with trace $T(u)|_{x=0} = u(0) = 0$. Show that $\frac{u(x)}{x} \in L^p(I)$ with

$$\left\| \frac{u(x)}{x} \right\|_{L^p(I)} \leqslant \frac{p}{p-1} \|u'\|_{L^p(I)}$$

where $u' = \frac{\mathrm{d}u}{\mathrm{d}x}$.

(c) Let $f \in L^2(I)$ such that $\frac{f(x)}{x} \in L^2(I)$. Consider the boundary value problem

$$\begin{cases} -\dfrac{\mathrm{d}^2 u}{\mathrm{d}x^2} + \dfrac{u(x)}{x^2} - \dfrac{\mathrm{d}u}{\mathrm{d}x} = \dfrac{f(x)}{x^2}\,, & \text{for } x \in I\,, \\ u(0) = 0\,, \quad u'(1) = 0\,. \end{cases} \tag{$\dagger$}$$

Set $H = \{v \in H^1(I) : T(v)|_{x=0} = v(0) = 0\}$.

(i) Show that H is a Hilbert space with the standard H^1–inner product.

(ii) Show that if $u \in H$ then $\frac{u(x)}{x} \in L^2(I)$.

(iii) Suppose there exists a function $u \in H^2(I)$ satisfying ($\dagger$) a.e. in I. Show that

$$\int_0^1 u'v'dx + \int_0^1 \frac{uv}{x^2}dx - \int_0^1 u'vdx = \int_0^1 \frac{fv}{x^2}dx \tag{$\dagger\dagger$}$$

for all $v \in H$.

(iv) Prove that there exists a unique $u \in H$ satisfying ($\dagger\dagger$).

[You may assume without proof the Trace and Lax Milgram Theorems, but you must show the Theorems apply.] [30]

3 Let $U \subset \mathbb{R}^3$ be open, and bounded with smooth boundary, and let $T > 0$ be fixed. Define $U_T := (0,T)\times U$, $\Sigma_t := \{t\}\times U$, $\partial^* U_T := [0,T]\times\partial U$. Given $\psi_0 \in H^1_0(U)$, $\psi_1 \in L^2(U)$ and $f \in L^2(U_T)$, consider the initial boundary value problem

$$\begin{cases} u_{tt} - \Delta u = f & \text{in } U_T\,, \\ u = \psi_0\,, u_t = \psi_1 & \text{on } \Sigma_0\,, \\ u = 0 & \text{on } \partial^* U_T\,. \end{cases} \qquad (\diamond)$$

Let $X_T = L^\infty((0,T); H^1_0(U)) \cap W^{1,\infty}((0,T); L^2(U))$ and equip this space with the norm

$$\|u\|_{X_T} := \|u\|_{L^\infty_t H^1_x} + \|u_t\|_{L^\infty_t L^2_x} = \operatorname*{ess\,sup}_{t\in(0,T)} \left(\|u(t,\cdot)\|_{H^1(\Sigma_t)} + \|u_t(t,\cdot)\|_{L^2(\Sigma_t)} \right) .$$

You may assume that a unique weak solution $u \in X_T$ to $(\diamond)$ exists satisfying

$$\|u\|_{X_T} \leqslant C_0 \left(\|\psi_0\|_{H^1(\Sigma_0)} + \|\psi_1\|_{L^2(\Sigma_0)} + \|f\|_{L^2(U_T)} \right) ,$$

for some constant C_0 depending only on U and T.

(a) Let $p_1, \ldots, p_n \in (0,\infty)$ such that $\sum_{i=1}^n \frac{1}{p_i} = 1$. Prove that for measurable functions $f_1, \ldots, f_n$ defined on U

$$\Big\| \prod_{i=1}^n f_i \Big\|_{L^1(U)} \leqslant \prod_{i=1}^n \Big\| f_i \Big\|_{L^{p_i}(U)}.$$

[You may assume the standard Hölder inequality for the product of two functions.]

(b) Let $w \in X_T$. Show that $w^2 \sin(w) \in L^2(U_T)$ with

$$\|w^2 \sin(w)\|_{L^2(U_T)} \leqslant \beta T^{\frac{1}{2}} \|w\|^2_{L^\infty_t H^1_x}$$

for some $\beta > 0$ depending only on U. If also $\tilde{w} \in X_T$, show that

$$\begin{aligned} &\|w^2 \sin(w) - \tilde{w}^2 \sin(\tilde{w})\|_{L^2(U_T)} \\ &\qquad \leqslant \gamma T^{\frac{1}{2}} \|w - \tilde{w}\|_{L^\infty_t H^1_x} \Big(\|w\|_{L^\infty_t H^1_x} + \|w\|^2_{L^\infty_t H^1_x} + \|\tilde{w}\|_{L^\infty_t H^1_x} + \|\tilde{w}\|^2_{L^\infty_t H^1_x} \Big) \end{aligned}$$

for some $\gamma > 0$ depending only on U.

(c) Fix $\psi_0 \in H^1_0(U)$ and $\psi_1 \in L^2(U)$. Let

$$X_{b,\tau} = \{u \in X_\tau : \|u\|_{X_\tau} \leqslant b\}.$$

Let A be the map which takes $w \in X_{b,\tau}$ to the unique weak solution $u \in X_\tau$ of $(\diamond)$ with f given by $f = w^2 \sin(w)$. Show that $A : X_{b,\tau} \to X_{b,\tau}$ is a contraction map provided $b > 0$ is sufficiently large and $0 < \tau < T$ is sufficiently small.

(d) Deduce that the semilinear wave equation:

$$\begin{cases} u_{tt} - \Delta u = u^2 \sin(u) & \text{in } U_\tau\,, \\ u = \psi_0\,, u_t = \psi_1 & \text{on } \Sigma_0\,, \\ u = 0 & \text{on } \partial^* U_\tau\,, \end{cases}$$

has a weak solution $u \in X_\tau$ provided $\tau \in (0,T)$ is sufficiently small. [30]

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