UNIVERSITY OF CAMBRIDGE

MAMA/102

MAT3

MATHEMATICAL TRIPOS Part III

Friday 31 May 2024 9:00 pm to 12:00 pm

PAPER 102

LIE ALGEBRAS AND THEIR REPRESENTATIONS

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

Attempt no more than **FOUR** questions.
There are **FIVE** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 (a) (i) Let $\varphi : \mathfrak{sl}_2(\mathbb{C}) \to \mathfrak{gl}(V)$ be a finite-dimensional representation of $\mathfrak{sl}_2(\mathbb{C})$. Define the *Casimir element*, $\Omega \in \mathfrak{gl}(V)$. You may assume that Ω commutes with $\varphi(x)$ for all $x \in \mathfrak{sl}_2(\mathbb{C})$.

State Schur's lemma. If V is irreducible, show that Ω acts on V by a scalar. Find this scalar in terms of the highest weight of V.

(ii) Let $\varphi : \mathfrak{sl}_2(\mathbb{C}) \to \mathfrak{gl}(V)$ be a finite-dimensional representation of $\mathfrak{sl}_2(\mathbb{C})$ and let W be a subrepresentation. Prove Weyl's theorem in the special case where W has codimension 1 in V.

(b) Let $a, b \in \mathbb{C}$ and define a vector space W over $\mathbb{C}$ with (countably infinite) basis

$$\ldots, w_{-2}, w_{-1}, w_0, w_1, w_2, \ldots$$

Define linear transformations h_W, f_W of W by the rules $h_W(w_i) = (a + 2i)w_i$ and $f_W(w_i) = w_{i-1}$, for all $i \in \mathbb{Z}$.

(i) Show that, subject to the constraint $e_W(w_0) = bw_1$, there is a unique way to define e_W so as to define an $\mathfrak{sl}_2(\mathbb{C})$-action on W.

(ii) Show that any non-zero $\mathfrak{sl}_2(\mathbb{C})$-subrepresentation of W contains w_i for some i.

(iii) Show that W is reducible (meaning that W is not irreducible) if and only if $b = ja + j(j+1)$ for some $j \in \mathbb{Z}$.

2 (a) Let $\varphi : \mathfrak{g} \to \mathfrak{gl}(V)$ be a finite-dimensional representation of the Lie algebra $\mathfrak{g}$. What is the *trace form* for V? Define the *Killing form*. What does it mean to say the form is $\mathfrak{g}$-invariant?

(b) Let $\mathfrak{g}$ be a finite-dimensional complex semisimple Lie algebra and $\mathfrak{t}$ a Cartan subalgebra. For each root $\alpha \in \mathfrak{t}^*$, prove the existence of an $\mathfrak{sl}_2$-subalgebra $\mathfrak{m}_\alpha = \langle e_\alpha, h_\alpha, f_\alpha \rangle$ with $e_\alpha \in \mathfrak{g}_\alpha, f_\alpha \in \mathfrak{g}_{-\alpha}$ and $h_\alpha \in \mathfrak{t}$ satisfying $[h_\alpha, e_\alpha] = 2e_\alpha$, $[h_\alpha, f_\alpha] = -2f_\alpha$ and $[e_\alpha, f_\alpha] = h_\alpha$. [You may assume that the restriction of the Killing form to $\mathfrak{t} \times \mathfrak{t}$ is non-degenerate. You may use Lie's theorem.]

(c) Let $\mathfrak{g}$ be a Lie algebra and V a finite-dimensional representation of $\mathfrak{g}$.

(i) Show that the vector space $\mathfrak{g} \oplus V$ is a Lie algebra under the Lie bracket

$$[(x, v), (x', v')] = ([x, x'], xv' - x'v),$$

for $x, x' \in \mathfrak{g}$, $v, v' \in V$. Denote this Lie algebra by $\mathfrak{g} \ltimes V$.

(ii) Using this construction, give an example of a Lie algebra $\mathfrak{h}$ such that the derived algebra $[\mathfrak{h}, \mathfrak{h}]$ equals $\mathfrak{h}$ and $Z(\mathfrak{h}) = \{0\}$ but $\mathfrak{h}$ is neither simple nor the direct product of simple Lie algebras.

(iii) Express the Killing form K on $\mathfrak{g} \ltimes V$ in terms of the Killing form κ on $\mathfrak{g}$. When is the Killing form K on $\mathfrak{g} \ltimes V$ non-degenerate? [Hint: you may wish to use the Cartan–Killing criterion.]

3

(a) (i) Let (Φ, E) be a root system. What is a *Weyl chamber*? What does it mean for a subset $\Delta \subseteq \Phi$ to be a *root basis*? Briefly describe the construction of root bases Δ_γ for certain elements γ (proofs are not required).

(ii) If α is positive but not simple, show that $\alpha - \beta$ is a root (necessarily positive) for some $\beta \in \Delta$. Deduce that every positive root can be written in the form $\alpha_1 + \cdots + \alpha_k$ ($\alpha_i \in \Delta$, not necessarily distinct) in such a way that each partial sum $\alpha_1 + \cdots + \alpha_i$ is a root.

Show also that if α is simple then the reflection w_α permutes the positive roots other than α.

(b) Consider the Lie algebra

$$\mathfrak{g} = \mathfrak{so}_6 = \{x \in \mathfrak{gl}_6(\mathbb{C}) : xJ + Jx^T = 0\},$$

where

$$J = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

Let $\mathfrak{t}$ be the space of diagonal matrices in $\mathfrak{g}$, and let Φ be the set of roots of $\mathfrak{g}$ with respect to $\mathfrak{t}$.

In parts (i), (ii), (iii), (iv) your answers can be brief.

(i) Explicitly describe the elements of Φ as elements of $\mathfrak{t}^*$.

(ii) Find a root basis $\Delta \subseteq \Phi$. Draw and label the Dynkin diagram of Φ.

(iii) For each element $\alpha_i \in \Delta$, explicitly describe the image of elements of Δ under the simple reflection w_{α_i}.

(iv) Describe an automorphism of Φ that is not given by an element of the Weyl group.

(v) Explain briefly why $\mathfrak{so}_6 \cong \mathfrak{sl}_4$. Results from the course can be quoted if they are clearly stated.

4 Let Φ be a root system in the inner product space E.

(i) Let $\alpha, \beta \in \Phi$ with $\beta \neq \pm\alpha$. Prove the finiteness lemma, namely (in the usual notation) that

$$\langle \alpha, \beta^\vee \rangle . \langle \beta, \alpha^\vee \rangle \in \{0, 1, 2, 3\}.$$

(ii) Show that the only rank 2 root systems are, up to isomorphism, types A_2, B_2, G_2 and $A_1 \times A_1$.

(iii) What does it mean for an irreducible root system Φ to be *simply laced*? Show that Φ is simply laced if and only if $\langle \alpha, \beta^\vee \rangle \in \{0, \pm 1\}$ for all α, β with $\beta \neq \pm\alpha$.

(iv) Show that if Φ is a root system and $\alpha, \beta \in \Phi$ are roots with $\alpha \neq \pm\beta$ then the restriction of $w_\alpha w_\beta$ to the span of α and β is a rotation, and determine the angle of this rotation. Deduce that the subgroup of the Weyl group W generated by w_α, w_β is a dihedral group with rotational subgroup generated by $w_\alpha w_\beta$. Hence, or otherwise, find the Weyl groups of the rank 2 root systems found in (ii).

5 (a) (i) Let $\mathfrak{g}$ be a complex semisimple Lie algebra and let $\mathfrak{t}$ be a Cartan subalgebra. Given $\lambda \in \mathfrak{t}^*$ briefly define the *Verma module* $M(\lambda)$ associated to λ and state its universal property. [You may assume the existence of the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$.]

Explain briefly why there exists a unique irreducible highest weight module $V = V(\lambda)$ with highest weight λ. State a condition for V to be finite-dimensional.

(ii) Let $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$. Take $\lambda = -d$ where $d > 0$. Show that $M(\lambda)$ is an infinite-dimensional irreducible $\mathfrak{g}$-module.

(b) State the Weyl character formula and the Weyl denominator formula, briefly defining the notation that you use.

Show that the character $\mathrm{ch}(V(k\rho)) = e^{k\rho} \prod_{\alpha \in \Phi^+} (1 + e^{-\alpha} + \cdots + e^{-k\alpha})$ for any $k \in \mathbb{N}$ and where ρ denotes half the sum of the positive roots.

(c) Let $\mathfrak{g} = \mathfrak{so}_5(\mathbb{C})$, and assume α_1 is a short root.

(i) Sketch the root system of type B_2, and indicate the fundamental dominant weights ω_1, ω_2. Let $\lambda = a\omega_1 + b\omega_2$ be a dominant weight. Using the Weyl dimension formula, find a formula for $\dim V(\lambda)$ in terms of a and b. [You are not asked to prove the dimension formula.]

(ii) Let V be the defining 5-dimensional representation of $\mathfrak{g}$. Explain briefly why $V \cong V(\omega_2)$. Explain briefly why $V(2\omega_2)$ is a subrepresentation of $V \otimes V$ and compute its dimension.

END OF PAPER