UNIVERSITY OF CAMBRIDGE

MAMA/101

MAT3

MATHEMATICAL TRIPOS Part III

Monday 3 June 2024 1:30 pm to 4:30 pm

PAPER 101

COMMUTATIVE ALGEBRA

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

Attempt no more than **FOUR** questions.
There are **FIVE** questions in total.
The questions carry equal weight.

The term *ring* stands for a commutative unital ring, and
the term *module* stands for a module over such a ring.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

i. Define each of the following module properties: Noetherian, Free, Flat, Projective.

[If we learned more than one characterization of a certain property, choose one.]

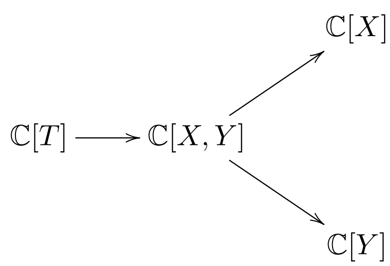
ii. Prove or disprove each of the following statements (for a ring R):

(a) If M is a finitely generated R-module and N is a noetherian R-module then $M \otimes_R N$ is a noetherian R-module.

(b) If A and B are nonzero algebras over a field k such that $A \otimes_k B$ is finitely generated as a k-algebra then each of A and B is finitely generated as a k-algebra.

(c) For R-modules M and N, if $M \otimes_R N \cong R^n$ for some integer $n \geqslant 1$ then M and N are projective R-modules.

(d) For every choice of three $\mathbb{C}$-algebra homomorphisms as in the diagram below



if $\mathbb{C}[X]$ and $\mathbb{C}[Y]$ are flat $\mathbb{C}[T]$-modules, then $\mathbb{C}[X] \otimes_{\mathbb{C}[X,Y]} \mathbb{C}[Y]$ is a flat $\mathbb{C}[T]$-module.

2

i. Let R be a ring such that every nonzero proper ideal of R is maximal. What is the maximal possible cardinality of $\operatorname{mspec} R$?

ii. Define the Jacobson radical $J(R)$ of a ring R.
Let $A \subset B$ be an integral extension of rings. Prove that $J(A) = J(B) \cap A$.

iii. Give an example of a nonzero module M over a ring R, and a positive integer k such that $M^{\otimes k}$ is the zero module, and prove that such an R-module M must not be finitely generated (justify your answer).

iv. Is every maximal ideal of $A = \mathbb{C}[T_1, T_2, \ldots]$ of the form $(T_1 - x_1, T_2 - x_2, \ldots)$, where $x_i \in \mathbb{C}$ for all $i \geqslant 1$?

[**Hint:** First prove that for a field extension L/K and an element $t \in L$, transcendental over K, and distinct $x_1, \ldots, x_n \in K$, the set $\{\frac{1}{t-x_1}, \ldots, \frac{1}{t-x_n}\}$ is linearly independent over K.]

3

i. Give an example of finite dimensional real vector spaces V, U, W and a non-injective $\mathbb{R}$-linear map $f\colon V \otimes_{\mathbb{R}} U \to W$ such that $f(v_1 \otimes u_1) \neq f(v_2 \otimes u_2)$ for all $v_1, v_2 \in V$ and $u_1, u_2 \in U$ such that $v_1 \otimes u_1 \neq v_2 \otimes u_2$. Briefly explain why your example has the required properties.

ii. Prove or disprove: If S is a multiplicative subset of an integral domain R, then for every subring A of $S^{-1}R$ there is a multiplicative subset T of R such that $A = T^{-1}R$ (where we view localizations of R as subrings of $\operatorname{Frac} R$ in the natural way).

iii. Let $F \subsetneq K$ be fields. Prove that the ring $K \otimes_F K$ is not a field.

iv. Let $(R, \mathfrak{m})$ be a noetherian local ring, and $x \in \mathfrak{m}$ not a zero divisor, such that $R/(x)$ is an integral domain. Prove that R is an integral domain.

4

i. State both the weak and strong versions of the nullstellensatz, and deduce the strong version from the weak version.

ii.

(a) Find $f \in \mathbb{R}[T_1, T_2]$ such that for $A = \mathbb{R}[T_1, T_2] / (f)$, $t_1 = T_1 + (f)$ and $t_2 = T_2 + (f)$, the contraction map $\operatorname{spec} A \to \operatorname{spec} \mathbb{R}[t_1]$ is not surjective.

(b) Let $f \in \mathbb{R}[T_1, \ldots, T_n]$, $\deg f > 0$, $A = \mathbb{R}[T_1, \ldots, T_n] / (f)$, and write $t_i = T_i + (f)$ for each $1 \leqslant i \leqslant n$.
Prove that there is a matrix $M \in M_{n \times n}(\mathbb{Z})$ such that (I) $\det M \neq 0$, (II) $\sup_{i,j} |M_{ij}| \leqslant \deg f$, and (III) for $\begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = M \begin{pmatrix} t_1 \\ \vdots \\ t_n \end{pmatrix}$, the contraction map $\operatorname{spec} A \to \operatorname{spec} \mathbb{R}[y_1, \ldots, y_{n-1}]$ is surjective.

iii. Let $A \subset B$ be an integral extension, and let $I \subset J$ be ideals of B such that $I \cap A = J \cap A$ and I is prime. Prove that $I = J$. [**Hint:** Use a similar claim from the lectures.]

iv. A *Jacobson ring* is a ring R such that for every ideal I of R, $\sqrt{I}$ is equal to the intersection of all maximal ideals of R containing I.
Let $R \subset S$ be an integral extension of rings such that R is a Jacobson ring. Prove that S is a Jacobson ring.

5

i. Define the *height* $\operatorname{ht} I$ of a proper ideal I of a ring R.

State and prove Krull's height theorem.

ii. State and prove the Hilbert–Serre Theorem.

iii. Assume that R is a noetherian ring of Krull dimension $d < \infty$. Let $0 \leqslant n \leqslant d$ be a non-negative integer such that $|\{\mathfrak{p} \in \operatorname{spec} R \mid \operatorname{ht} \mathfrak{p} = n\}| < \infty$. Prove that $n \in \{0, d\}$.

END OF PAPER