UNIVERSITY OF CAMBRIDGE

MAMA/354, NST3AS/354, MAAS/354

MAT3

MATHEMATICAL TRIPOS Part III

Tuesday, 13 June, 2023 9:00 am to 11:00 am

PAPER 354

GAUGE/GRAVITY DUALITY

Before you begin please read these instructions carefully

Candidates have TWO HOURS to complete the written examination.

Attempt **BOTH** questions.
There are **TWO** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(a) Suppose that there exists a (non-supersymmetric) holographic theory of quantum gravity in AdS_4 for which the only low energy bulk field is the *graviton*.

(Assume that all other bulk fields have a mass $M \gg 1/L_{\mathrm{AdS}}$ where L_{AdS} is the AdS scale, and that $\ell_{\mathrm{Planck}} \ll L_{\mathrm{AdS}}$. Assume also that the graviton has reflecting boundary conditions at spatial infinity, satisfying the unitary bound.)

Consider the dual CFT in an Einstein universe $S^2 \times \mathbb{R}$, where the S^2 has radius R. Determine the following for each of the lowest 3 distinct energy levels $E_0 < E_1 < E_2$ of the CFT:

(i) The energy E_n.

(ii) The degeneracy N_n.

(iii) The angular momenta j, in the sense of representations of $SU(2)$, the group of rotations of the S^2.

(iv) The number of gravitons in the AdS bulk.

[Hint: use the operator-state correspondence.]

(b) Consider a scalar primary operator $\mathcal{O}$ of dimension $\Delta > (d-2)/2$, in a CFT of dimension d. Consider the following 2-point function in Minkoswki spacetime:

$$F = \langle \mathcal{O}(\mathbf{x}_1)\, \Box \mathcal{O}(\mathbf{x}_2) \rangle \,,$$

where $\Box = \vec{\nabla}^2 - (\nabla_t)^2$ is the d'Alembertian operator. Let F_0 be the value of this correlator in the vacuum state.

(i) Using symmetry, determine F_0 at spacelike separation (up to a real multiplicative constant).

(ii) Write a formula for the (time-ordered) correlator F_0 in terms of the partition function $Z[J]$ of the field theory with source J, where J is conjugate to $\mathcal{O}$. Suppose that J is a nonzero constant. What is a necessary condition on Δ for the resulting theory to be a CFT for all constant values of J?

(iii) Assume now that the CFT is holographic and $\mathcal{O}$ is single-trace. Let z be the holographic coordinate in the usual AdS-Poincaré metric:

$$ds^2 = \frac{1}{z^2}\left(dz^2 + \eta_{ij}\, dx^i\, dx^j\right).$$

Write down a formula for J and $\mathcal{O}$ in terms of the dual bulk field $\phi(\mathbf{x}, z)$ (ignoring multiplicative constants). In the special case where $d = 3$, $\Delta = 1$, $J = 0$, write down a formula for F in a general state, in terms of $\phi(\mathbf{x}, z)$. Use the bulk equations of motion to eliminate any explicit derivatives with respect to the $\mathbf{x}_1$ or $\mathbf{x}_2$ directions.

2

An AdS-Schwarzschild black hole in 5 spacetime dimensions has the metric:

$$ds^2 = -f dt^2 + \frac{dr^2}{f} + r^2 d\Omega^2_{(3)}, \qquad f = 1 + \frac{r^2}{L^2} - \frac{\mu}{r^2}$$

where L is the AdS radius and μ is a parameter that is proportional to the mass of the black hole (relative to empty AdS).

(a) Derive the following formulas:

(i) Show that the radius of the horizon is given by:

$$r_H^2 = \frac{L^2}{2}\left(\sqrt{1 + \frac{4\mu}{L^2}} - 1\right).$$

(ii) Show that the inverse temperature (needed to avoid a conical singularity at $r = r_H$ in Euclidean signature) is given by

$$\beta_H = \frac{2\pi L^2 r_H}{2r_H^2 + L^2}.$$

(b) (i) By introducing a conical singularity at the horizon, show that the entropy S_{BH} of the black hole is equal to $A/4G$ where $A = 2\pi^2 r_H^3$. You may use the formula $S_{\mathrm{BH}} = (1 - \beta\partial_\beta)(-I_{\mathrm{grav}})$, where the Euclidean action is given by:

$$I_{\mathrm{grav}} = -\frac{1}{16\pi G_N}\int_{\mathcal{M}} d^5\mathbf{x}\,\sqrt{g}(R - 2\Lambda) + \int_{\partial\mathcal{M}} d^4\mathbf{x}\,[\text{boundary terms}],$$

where $\mathcal{M}$ is the Euclidean spacetime manifold, G_N is Newton's constant, $\Lambda = -6/L^2$, and the boundary terms include the Gibbons-Hawking term and holographic counterterms. Explain why neither these boundary terms nor Λ contribute to your calculation of S_{BH}.

(ii) Suppose that you instead calculate S_{BH} by varying the periodicity β at spatial infinity, and find the *smooth* solution to the Einstein equations in the bulk interior. Without doing the calculation, explain why this would give the same answer as the previous part.

(c) Consider the case of an eternal black hole with two asymptotically AdS regions, on the left and the right.

(i) Explain in detail what the quantities β and S_{BH} are dual to on the boundary, in terms of the wavefunction Ψ of the $\mathrm{CFT}_L \times \mathrm{CFT}_R$ system, and the energy E. Show on the CFT side that, for any first-order variation $\delta\rho$ of the state ρ of CFT_R, the variation satisfies the *first law of entanglement*: $\delta S = \beta\,\delta\langle E\rangle_\rho$.

(ii) Suppose now that a spherical shell of coherent scalar radiation is dropped into this black hole from the boundary $r = +\infty$, at time $t = 0$. (This may be implemented by means of a unitary transformation of the CFT state at time $t = 0$.) At late time, the system settles into a new black hole with mass $\mu' > \mu$. Where is the holographic entanglement entropy surface located in the resulting spacetime, and what is the associated holographic entropy? What is the significance of the area of the horizon at late times, from a thermodynamical viewpoint?

END OF PAPER