UNIVERSITY OF CAMBRIDGE

MAMA/352, NST3AS/352, MAAS/352

MAT3

MATHEMATICAL TRIPOS Part III

Wednesday, 7 June, 2023 1:30 pm to 3:30 pm

PAPER 352

NON-NEWTONIAN FLUID MECHANICS

Before you begin please read these instructions carefully

Candidates have TWO HOURS to complete the written examination.

Attempt **BOTH** questions.
There are **TWO** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

A constant-thickness layer of Bingham fluid is deposited around a cylindrical fibre of radius a and length L; using cylindrical coordinates, the fluid domain is thus written as $\{a \leqslant r \leqslant b,\ 0 \leqslant z \leqslant L\}$. The fibre is stationary and the surface of the fluid at $r = b$ is a free surface, assumed to remain of cylindrical shape. A pressure drop Δp is applied along the length of the fibre, such that the fluid flows in the positive z direction.

(a) What family of fluids does the Bingham fluid belong to? Give an example of complex-fluid phenomenology that is not captured by the Bingham fluid model. Give the full constitutive relationship for unidirectional flow of the Bingham fluid using the constitutive parameters σ_y and η.

(b) Assuming unidirectional flow, solve the inertialess Cauchy equations and determine the value of τ_{rz} in the fluid. Show that the hydrodynamic force exerted on the fibre is independent of σ_y.

(c) Examining the distribution of shear stresses in the fluid, determine the condition on Δp for flow to occur. Assuming that condition to be satisfied, solve for the flow and determine the speed at which the unyielded portion of the fluid moves (you do not need to solve for the location of the yield surface explicitly).

(d) The yielded region of the fluid is measured experimentally to experience normal stress differences. This is modelled by assuming a second-order fluid constitutive relationship for stress above the yield stress, with normal stress coefficients Ψ_1 and Ψ_2. By analogy with the behaviour of a second-order fluid in a steady shear flow, which component(s) of the normal stress do you expect will be non-zero? Show that they lead to a non-zero pressure gradient in the radial direction and determine its sign and its dependence on Ψ_1 and Ψ_2.

Hint: The inertialess Cauchy equations and the components of the shear rate tensor in cylindrical coordinates are

$$\begin{aligned}
\frac{\partial p}{\partial r} &= \frac{1}{r}\frac{\partial (r\tau_{rr})}{\partial r} + \frac{1}{r}\frac{\partial \tau_{r\theta}}{\partial \theta} - \frac{\tau_{\theta\theta}}{r} + \frac{\partial \tau_{rz}}{\partial z},\\
\frac{1}{r}\frac{\partial p}{\partial \theta} &= \frac{1}{r^2}\frac{\partial (r^2\tau_{r\theta})}{\partial r} + \frac{1}{r}\frac{\partial \tau_{\theta\theta}}{\partial \theta} + \frac{\partial \tau_{\theta z}}{\partial z},\\
\frac{\partial p}{\partial z} &= \frac{1}{r}\frac{\partial (r\tau_{rz})}{\partial r} + \frac{1}{r}\frac{\partial \tau_{\theta z}}{\partial \theta} + \frac{\partial \tau_{zz}}{\partial z}.
\end{aligned}$$

$$\dot\gamma_{rr} = 2\frac{\partial u_r}{\partial r}, \quad \dot\gamma_{r\theta} = \frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}, \quad \dot\gamma_{rz} = \frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r},$$

$$\dot\gamma_{\theta\theta} = 2\left(\frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r}\right), \quad \dot\gamma_{\theta z} = \frac{1}{r}\frac{\partial u_z}{\partial \theta} + \frac{\partial u_\theta}{\partial z}, \quad \dot\gamma_{zz} = 2\frac{\partial u_z}{\partial z}.$$

2

The constitutive relationship for the Johnson–Segalman–Oldroyd fluid relates the deviatoric stress tensor in the fluid, $\boldsymbol{\tau}$, to the shear rate tensor, $\dot{\boldsymbol{\gamma}}$, as

$$\boldsymbol{\tau} + \lambda_1 \overset{\square}{\boldsymbol{\tau}} = \eta \left(\dot{\boldsymbol{\gamma}} + \lambda_2 \overset{\square}{\dot{\boldsymbol{\gamma}}} \right), \qquad (\dagger)$$

where λ_1, λ_2 and η are positive constants and where the Gordon–Schowalter convected derivative for a tensor $\boldsymbol{\sigma}$ is an objective derivative defined as

$$\overset{\square}{\boldsymbol{\sigma}} \equiv \overset{\triangledown}{\boldsymbol{\sigma}} + \frac{a}{2}(\dot{\boldsymbol{\gamma}} \cdot \boldsymbol{\sigma} + \boldsymbol{\sigma} \cdot \dot{\boldsymbol{\gamma}}),$$

where a is a small, dimensionless parameter ($a > 0$).

(a) What is meant by the statement that the derivative is *objective*? Give another example of an objective derivative and give an example that is not objective. What are the physical interpretations of the parameters η, λ_1 and λ_2? Propose simple experiments to estimate their values.

(b) The fluid characterised by the constitutive relationship in Eq. ($\dagger$) undergoes uniform small-amplitude oscillatory shear of the form $\dot{\gamma}(t) = \dot{\gamma}_0 \cos \omega t$. What are the conditions on $\dot{\gamma}_0$ required to be allowed to linearise Eq. ($\dagger$)? Compute the linearised oscillatory stress response and deduce the values of the storage modulus $G'(\omega)$ and the loss modulus $G''(\omega)$.

(c) Returning to the full nonlinear constitutive relationship in Eq. ($\dagger$), the fluid now undergoes steady two-dimensional shear with constant shear rate $\dot{\gamma}$. Assuming $\tau_{xz} = \tau_{yz} = 0$, calculate the shear viscosity for this fluid. What are the conditions on the values of λ_1 and λ_2 for the fluid to be shear-thinning? [*Hint: You may introduce the notation* $W_1 \equiv \lambda_1 \dot{\gamma}$, $W_2 \equiv \lambda_2 \dot{\gamma}$ *to simplify your algebra.*]

END OF PAPER