MAMA/347, NST3AS/347, MAAS/347

MAT3

MATHEMATICAL TRIPOS Part III

Thursday, 8 June, 2023 1:30pm to 3:30pm

PAPER 347

ASTROPHYSICAL BLACK HOLES

Before you begin please read these instructions carefully

Candidates have TWO HOURS to complete the written examination.

Attempt no more than **TWO** questions.
There are **THREE** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 (a) Consider a steady, spherically symmetric accretion of gas onto a supermassive black hole with mass, M. You may assume that the gas is isothermal and at rest at infinity. From mass conservation and momentum equations show that there is a sonic transition in the flow of interest. Hence, derive the expression for the mass accretion rate, $\dot{M}$, as a function of M, gas density and sound speed at infinity.

(b) Comment on which astrophysical situations and for which type of galaxy this mode of accretion flow could apply and why. List and discuss at least five reasons that could invalidate the assumptions inherent to this accretion flow.

(c) When considering 'Bondi-like' accretion flows on a supermassive black hole in the presence of a galactic potential the following simple model may be adopted,

$$\dot{M} = \frac{\pi G^2 M_{\rm enc}^2\, \rho_\infty}{(c_{\rm s,\infty}^2 + \sigma^2)^{\frac{3}{2}}}\,, \qquad (1)$$

where $\dot{M}$ is the mass accretion rate, $M_{\rm enc}$ is the enclosed mass, ρ_∞ is the gas density at infinity, $c_{\rm s,\infty}$ is the gas sound speed at infinity and σ is the velocity dispersion. Derive the limiting cases of this equation in the regimes where: *i)* $c_{\rm s,\infty} \gg \sigma$ and $M_{\rm enc} \to M$, where M is the mass of the black hole; *ii)* $\sigma \gg c_{\rm s,\infty}$, and discuss their physical meaning.

(d)

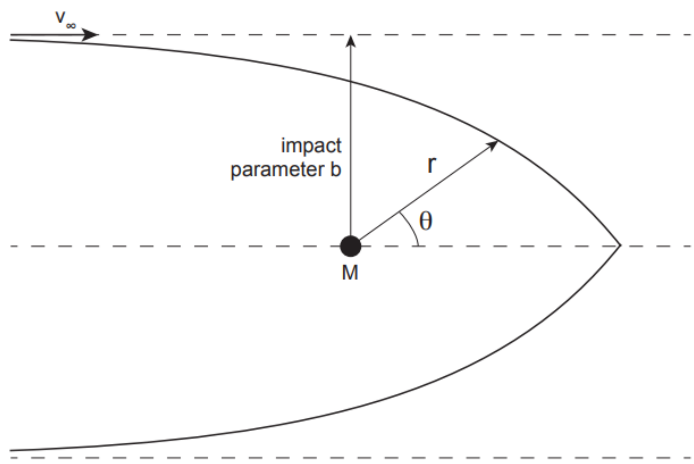

Figure 1: Simple sketch of a supersonic motion of a black hole through a uniform density gas cloud where two fluid streamlines with impact parameter b are illustrated.

[QUESTION CONTINUES ON THE NEXT PAGE]

Consider a black hole, with mass M, moving supersonically with a velocity v_∞ through a gas cloud with a uniform density ρ_∞. You may assume that the fluid streamlines follow essentially ballistic trajectories in the gravitational field of the black hole as illustrated in Figure 1. The equations of motion of this problem are:

$$\ddot{r} - r\dot{\theta}^2 = -\frac{GM}{r^2}\,, \qquad r^2\dot{\theta} = bv_\infty\,, \tag{2}$$

where r and θ are cylindrical polar coordinates, G is the gravitational constant and b is the impact parameter. By appropriately manipulating equation (2) show that the solution can be written as follows

$$r^{-1} = c_1\cos\theta + c_2\sin\theta + \frac{GM}{(bv_\infty)^2}\,, \tag{3}$$

where c_1 and c_2 are two constants that you should calculate using the boundary conditions of the problem.

Hence, determine the distance from the black hole where the two streamlines each with the impact parameter b, as in Figure 1, collide and lead to cancellation of the azimuthal velocity.

From this derive the critical impact parameter, $b_{\rm crit}$, within which the gas will be bound to the black hole and calculate the mass accretion rate onto the black hole. Comment how this differs from the 'standard' Bondi solution.

2 (a) Consider a steady, geometrically thin and optically thick α accretion disc around a supermassive black hole, with a mass M. Assuming that black body radiation applies at each annulus at distance R in the disc, write down how the black body temperature of the disc, $T_{\rm BB}(R)$, is determined by the viscous dissipation rate, $F_{\rm diss}$.

Recalling that $F_{\rm diss} = \nu\Sigma R^2\left(\frac{d\Omega}{dR}\right)^2$, where ν is the kinematic viscosity, Σ is the disc surface density and Ω is the angular velocity, derive how $T_{\rm BB}(R)$ depends on the mass accretion rate, $\dot{m}$, M and R.

Furthermore, derive that the black body spectrum integrated over all radii, $S_{\bar{\nu}}$, for an intermediate range of frequencies $\bar{\nu}$, scales as $\bar{\nu}^{\frac{1}{3}}$, with $k_{\rm B}T_{\rm out} \ll h\bar{\nu} \ll k_{\rm B}T_{\rm in}$, where $k_{\rm B}$ is the Boltzmann constant and h is the Planck constant. $T_{\rm in}$ and $T_{\rm out}$ are disc black body temperatures in the innermost region of the disc and in the outer disc, respectively.

(b) For an α accretion disc around a supermassive black hole, qualitatively explain different regions that may exist in the disc which are dominated by radiation pressure, gas pressure and different opacities.

For the region of the disc dominated by radiation pressure calculate the total cooling rate and the total heating rate to determine if this region is thermally unstable or not. Recall that for gas to be thermally unstable the following needs to be satisfied:

$$\frac{\partial \dot{Q}}{\partial T_{\rm c}} < 0\,, \tag{1}$$

where $\dot{Q}$ is the *net* cooling rate and $T_{\rm c}$ is the mid-plane disc temperature. Qualitatively discuss your findings in the context of observed AGN variability.

(c) For the standard, steady α disc, vertically integrating the toroidal component of the Navier-Stokes equation one obtains:

$$\Sigma R u_{\rm R}\frac{d(R^2\Omega)}{dR} = \frac{d}{dR}\left(\nu\Sigma R^3\frac{d\Omega}{dR}\right), \tag{2}$$

where $u_{\rm R}$ is the radial velocity. Show that radially integrating this equation from the ISCO to some large R one can estimate $u_{\rm R}$ to be:

$$u_{\rm R} \approx \alpha\frac{H^2}{R^2}\frac{u_\phi l}{l - l_{\rm ISCO}}\,, \tag{3}$$

where H is the vertical height of the disc, u_ϕ is the azimuthal velocity and l is the specific angular momentum.

Assuming that $u_{\rm R}$ undergoes a sonic transition close to ISCO and that the disc at the sonic transition is still geometrically thin, show that for $\alpha \ll 1$ the no torque boundary condition directly implies that the gas specific angular momentum is conserved as it crosses ISCO.

3 (a) Explain what the maximal radiative efficiency is, and how it can be simply defined in the Newtonian limit. How does it affect the accretion rate onto a supermassive black hole? Why and how may it depend on the spin of the black hole? What is the difference between radiative efficiency and maximal radiative efficiency? Describe two examples of radiatively inefficient flows.

(b) Consider an optically thin, adiabatic Bondi accretion flow onto a supermassive black hole of mass M and luminosity L. Assuming that the gas density and sound speed at infinity are ρ_∞ and $c_{\mathrm{s},\infty}$, respectively, derive that the radiative efficiency is a function of L and Eddington luminosity only, with a constant prefactor of 9×10^{-3}. Provide a physical interpretation of this result.

(c) Explain qualitatively what is meant by Soltan's argument and how from observations we can constrain the average radiative efficiency of a cosmological population of AGN. Based on these considerations, discuss which types of accretion flows are most likely to fuel AGN and why.

(d) Consider now a steady, geometrically thick accretion disc, with disc height H comparable to the disc radius R. Write down the expression for the mass accretion rate through this disc, $\dot{M}$.

If the disc density is low enough the electron-proton thermal equilibration timescale, $t_{\mathrm{e-p}}$, will be longer than the gas inflow timescale. The electron-proton thermal equilibration timescale applicable to this problem can be approximated as

$$t_{\mathrm{e-p}} \approx 0.06 \frac{1}{n \sigma_{\mathrm{T}} c} \frac{m_{\mathrm{p}}}{m_{\mathrm{e}}} \left(\frac{k_{\mathrm{B}} T_{\mathrm{e}}}{m_{\mathrm{e}} c^2} \right)^{\frac{3}{2}} , \qquad (1)$$

where n is the gas number density, σ_{T} is the Thompson cross-section, c is the speed of light, m_{p} is the proton mass, m_{e} is the electron mass, k_{B} is the Boltzmann constant and T_{e} is the electron temperature.

In this case show that

$$\frac{\dot{M}}{\dot{M}_{\mathrm{Edd}}} \lesssim 0.4 \alpha^2 , \qquad (2)$$

where $\dot{M}_{\mathrm{Edd}}$ is the Eddington accretion rate and α is the Shakura-Sunayev parameter. You may assume that the electrons are mildly relativistic.

Qualitatively explain how the radiative efficiency of this flow compares to the canonical value of 0.1 adopted for thin α accretion discs.

[*Hint: To derive this results you may use analogous arguments to the ones used to estimate the characteristic timescales of a thin α disc.*]

END OF PAPER