MAMA/342, NST3AS/342, MAAS/342, NST3PHY/TQM, MAPY/TQM

MAT3

MATHEMATICAL TRIPOS Part III

Monday, 12 June, 2023 9:00 am to 11:00 am

PAPER 342

TOPOLOGICAL QUANTUM MATTER

Before you begin please read these instructions carefully

Candidates have TWO HOURS to complete the written examination.

Attempt no more than **TWO** questions.
There are **THREE** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Consider a two-dimensional electron system described by the Chern-Simons (CS) Lagrangian density

$$\mathcal{L} = \frac{1}{4\pi\hbar} K_{IJ} \varepsilon^{\alpha\beta\gamma} a_{I\alpha} \partial_\beta a_{J\gamma},$$

where K_{IJ} is an $n \times n$ symmetric invertible matrix of integers and $a_{I\alpha}$ are CS gauge fields. Suppose that electrons (i.e., the system's microscopic constituents) have particle 3-current

$$J^\alpha = \frac{1}{2\pi\hbar} t_I \epsilon^{\alpha\beta\gamma} \partial_\beta a_{I\gamma},$$

where $\mathbf{t}$ is an n-component vector with $t_I = 1$ for all I; the number N of electrons in an area Ω is thus $N = \int_\Omega d^2x\, J^0$. (In both $\mathcal{L}$ and J^α we sum over repeated indices.)

(a) Add source terms to $\mathcal{L}$ and use the classical equations of motion (EOMs) for $a_{I\alpha}$ to show that encircling a quasiparticle of charges q_I (under $a_{I\alpha}$) with one of charges q'_I yields phase $\exp(2i\theta_{q'q})$ with $\theta_{q'q} = \pi \mathbf{q}' \cdot K^{-1}\mathbf{q}$, where $(\mathbf{q}^{(\prime)})_J = q_J^{(\prime)}$ for the n-component vector $\mathbf{q}^{(\prime)}$. Using the EOMs, also obtain the number N of electrons in a quasiparticle of charges $\mathbf{q}$. Using these results, show that for $K = \mathbb{1}_n$ quasiparticles with $\mathbf{q} = \mathbf{e}_J$, where $(\mathbf{e}_J)_I = \delta_{IJ}$, can be interpreted as electrons. [In obtaining $\theta_{q'q}$, you may use the shortcut of substituting the EOMs back to $\mathcal{L}$.]

The rest of the question focuses on $K = \mathbb{1}_n + pM$ with p integer and $M_{IJ} = 1$ for all $I, J = 1, \ldots, n$.

(b) Show that, compared with part (a), an additional CS flux $-2\pi\hbar p/(1+np)$ in all the n gauge fields is attached to quasiparticles with $\mathbf{q} = \mathbf{e}_J$. [In obtaining K^{-1}, you may want to use that $M^2 = nM$.]

(c) Show that quasiparticles with $\mathbf{q} = \mathbf{e}_J$ have constituent particle number $N = (1+np)^{-1}$. Obtain the allowed set of charges $\mathbf{q}$ an electron can have by requiring that (i) an electron encircling a quasiparticle with any $\mathbf{q}'$ yield phase $\exp(2i\theta_{q'q}) = 1$, (ii) be a fermion, and (iii) have unit constituent particle number N. Hence show that p must be even. [You may use that $q_I^{(\prime)}$ are integers.]

(d) Suppose that the process $T_\gamma(\mathbf{q})$ of creating a quasiparticle of charge $\mathbf{q}$ together with its antiparticle, dragging the quasiparticle along a path γ, and then reannihilating with its antiparticle is a unitary operation taking ground state to ground state. Working on the torus, use $T_\gamma(\mathbf{q})$, with suitable $\mathbf{q}$ and pairs of paths γ, to show that $\theta_{q'q} = \pi \mathbf{q}' \cdot K^{-1}\mathbf{q}$ implies ground state degeneracy of (at least) $|\det K|$ and that this equals $|1+np|$.

2 This question considers phase-flip and bit-flip codes, the relation of their concatenation to the surface code, and the corresponding error-correction thresholds.

(a) Consider the unperturbed one-dimensional Ising model

$$H = -J \sum_{j=1}^{n-1} X_j X_{j+1}, \quad J > 0.$$

By establishing stabilizer generators and logical operators, show that the ground space is the code space of an n-qubit code to detect phase-flip errors Z_j. Denoting this "phase-flip" code by C_n^Z, use the Knill-Laflamme conditions to establish the maximum number t of phase-flip errors that C_n^Z can correct, assuming that at most t phase flips may occur on any combination of qubits. Explain why C_n^Z cannot correct bit-flip errors X_j.

(b) Suppose that the probability of a phase-flip error on a qubit is p (with $0 \leqslant p < 1/2$), independently for each qubit. Show that maximum likelihood decoding of C_n^Z amounts to a majority vote. Obtain the corresponding error-correction threshold using the properties of the binomial distribution $B(n,p)$, including its mean np and standard deviation $\sqrt{np(1-p)}$.

(c) The n-qubit bit-flip code C_n^X is obtained from C_n^Z upon $X_j \leftrightarrow Z_j$ in all operators. Consider the concatenation $C_n^Z \circ C_m^X$ of C_n^Z with C_m^X, obtained by replacing the constituent qubits of C_n^Z by the logical qubits of C_m^X. Show that $C_n^Z \circ C_m^X$ can be viewed as a surface code on a lattice defined using n spheres joined south-pole-to-north-pole, such that the spheres' touching points define the vertices and, for each sphere, m north-pole-to-south-pole lines of longitudes define the links. Write down the logical operators, interpret them geometrically, and obtain the code distance.

(d) Consider $C_n^Z \circ C_m^X$ under the error model in part (b). Show that the error-correction threshold is the same as in part (b), however, the larger m the slower the logical error probability decays with n. [You may want to use that $((1-p) \pm p)^l = \Pr(\text{even}) \pm \Pr(\text{odd})$ for the probabilities $\Pr(\text{even})$ of an even and $\Pr(\text{odd})$ of an odd number of successes from the binomial distribution $B(l,p)$.]

3 This question studies aspects of Majorana zero modes.

(a) The Bogoliubov-de Gennes Hamiltonian for a one-dimensional superconductor reads

$$H_{\mathrm{BdG}} = \begin{pmatrix} \frac{p^2}{2m} - \mu & \Delta p \\ \Delta p & \mu - \frac{p^2}{2m} \end{pmatrix}, \quad \Delta, m > 0,$$

with $p = -i\hbar\partial_x$. Argue that a system with a superconductor for $x < 0$ and the vacuum for $x > 0$ satisfies $\mathrm{sgn}(\mu) = -\mathrm{sgn}(x)$. Use a long-wavelength approximation, describing also its domain of validity, to show that such a system hosts a zero-energy bound state ψ at the superconductor–vacuum interface. Express the characteristic spatial width of ψ using the parameters of H_{BdG}. Express the operator γ corresponding to ψ using fermion creation and annihilation operators and show that γ can be chosen Hermitian.

(b) Define Majorana zero modes (MZMs). Define adiabaticity in the context of MZM exchange and show, on general grounds, that adiabatically exchanging MZMs γ_a and γ_b implements $R_{ab}^{\pm} = \exp\left(\pm\frac{\pi}{4}\gamma_a\gamma_b\right)$ up to a phase. Interpret the sign $\pm$ in $R_{ab}^{\pm}$.

(c) Consider a system of $N+1$ MZMs γ_j $(j = 0, \ldots, N)$. Show that $\Pi_{\pm}^{(ab)} = \frac{1}{2}(\mathbb{1} \pm i\gamma_a\gamma_b)$ is a projector and that it describes a fermion-parity measurement. Suppose that the system is prepared in the $+1$ eigenstate of $i\gamma_N\gamma_0$. Show that, for this state, the sequence $\Pi_{+}^{(N0)}\Pi_{s_a}^{(a0)}\Pi_{s_b}^{(b0)}$ of measurements, with $\Pi_{s_j}^{(j0)} = (\mathbb{1} + is_j\gamma_j\gamma_0)/2$ and outcome $s_j = \pm1$, implements $R_{ab}^{s_a s_b}$ for $0 < a, b < N$.

(d) To achieve the desired braid, the measurements of $i\gamma_j\gamma_0$ in part (c) must yield specified outcomes s_j. Focusing on $\Pi_{s_b}^{(b0)}$, show that each of $s_b = \pm1$ occurs with 1/2 probability. Then show that a desired s_b can be achieved via "forced measurements" and briefly describe how the approach can be continued for the entire $\Pi_{+}^{(N0)}\Pi_{s_a}^{(a0)}\Pi_{s_b}^{(b0)}$ sequence.

END OF PAPER