UNIVERSITY OF CAMBRIDGE

MAMA/341, NST3AS/341, MAAS/341

MAT3

MATHEMATICAL TRIPOS Part III

Thursday, 1 June, 2023 1:30 am to 4:30 pm

PAPER 341

NUMERICAL SOLUTION OF DIFFERENTIAL EQUATIONS

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

Attempt **THREE** questions from Section A and **ONE** question from Section B.
Each question from Section B carries twice the weight of a question from Section A.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION A

1

Consider the ODE

$$\mathbf{y}' = A(\mathbf{y})\mathbf{y}, \quad t \geqslant 0, \qquad \mathbf{y}(0) = \mathbf{y}_0 \in \mathbb{R}^d,$$

where A is a $d \times d$ real-valued matrix function such that $A(\mathbf{x}) + A^\top(\mathbf{x}) = O$ for every $\mathbf{x} \in \mathbb{R}^d$.

a. Prove that $\|\mathbf{y}(t)\| = \|\mathbf{y}_0\|$ for all $t \geqslant 0$, where $\|\cdot\|$ is the standard Euclidean norm, $\|\mathbf{x}\| = \sqrt{\mathbf{x}^\top \mathbf{x}}$.

b. The ODE is solved with the implicit midpoint rule, which for $\mathbf{y}' = \mathbf{f}(\mathbf{y})$ reads

$$\mathbf{y}_{n+1} = \mathbf{y}_n + h\mathbf{f}\left(\frac{\mathbf{y}_n + \mathbf{y}_{n+1}}{2}\right).$$

Prove that $\|\mathbf{y}_n\| = \|\mathbf{y}_0\|$ for all $n \in \mathbb{Z}_+$.

c. Instead of the implicit midpoint rule, we solve the ODE with the trapezoidal rule,

$$\mathbf{y}_{n+1} = \mathbf{y}_n + \frac{h}{2}[\mathbf{f}(\mathbf{y}_n) + \mathbf{f}(\mathbf{y}_{n+1})].$$

Prove that

$$\|\mathbf{y}_{n+1}\|^2 - \|\mathbf{y}_n\|^2 = \frac{h}{2}\mathbf{y}_n^\top (A_{n+1} - A_n)\mathbf{y}_{n+1}.$$

2

The ODE $\mathbf{y}' = \mathbf{f}(\mathbf{y})$ is solved by the two-step method

$$\mathbf{y}_{n+2} - (1+a)\mathbf{y}_{n+1} + a\mathbf{y}_n = \frac{h}{12}[(5+a)\mathbf{f}(\mathbf{y}_{n+2}) + 8(1-a)\mathbf{f}(\mathbf{y}_{n+1}) - (1+5a)\mathbf{f}(\mathbf{y}_n)],$$

where a is a real parameter.

a. What is the order of the method?

b. For which values of a is the method convergent?

c. For which values of a is the method A-stable?

You should quote carefully all results from the lecture course you are using in your answer.

3

The Poisson equation

$$\Delta u = f, \qquad u = u(x_1, x_2, x_3),$$

is given in the three-dimensional cube $0 \leqslant x_1, x_2, x_3 \leqslant 1$ with zero Dirichlet boundary conditions. It is solved by the finite-difference method

$$u_{k-1,\ell,m} + u_{k+1,\ell,m} + u_{k,\ell-1,m} + u_{k,\ell+1,m} + u_{k,\ell,m-1} + u_{k,\ell,m+1} - 6u_{k,\ell,m} = (\Delta x)^2 f_{k,\ell,m}$$

for $k, \ell, m = 1, \ldots, n$, where $\Delta x = 1/(n+1)$, $u_{k,\ell,m} \approx u(k\Delta x, \ell\Delta x, m\Delta x)$ and $f_{k,\ell,m} \approx f(k\Delta x, \ell\Delta x, m\Delta x)$.

a. Find the order of the approximation, i.e. integer $p \geqslant 1$ such that

$$u_{k,\ell,m} = u(k\Delta x, \ell\Delta x, m\Delta x) + O((\Delta x)^{p+1}), \qquad k, \ell, m = 1, \ldots, n$$

for $n \gg 1$ and all suitably smooth f.

b The n^3 linear algebraic equations are written as $A\mathbf{u} = \mathbf{b}$, where $\mathbf{u} \in \mathbb{R}^{n^3}$ corresponds to an arbitrary ordering of the $u_{k,\ell,m}$s. Prove that A is a symmetric matrix.

c. Prove that A is negative definite, consequently that the system is nonsingular.

4

Consider the time-dependent PDE

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \alpha \frac{\partial u}{\partial x}, \qquad t \geqslant 0, \quad x \in [-1, 1],$$

where $u = u(x,t)$, while α is a real constant, given with an initial condition $u(x,0) = u_0(x)$ and zero boundary conditions.

a. Determine conditions on α so that the solution is well posed in the L_2 norm.

b. The PDE is solved by the semi-discretisation

$$u'_m = \frac{1}{(\Delta x)^2}(u_{m-1} - 2u_m + u_{m+1}) + \frac{\alpha}{2\Delta x}(u_{m+1} - u_{m-1}), \qquad m = -N, \ldots, N,$$

where $\Delta x = 1/(N+1)$ and $u_m(t) \approx u(m\Delta x, t)$. Is the method convergent?

5

a. Define what is meant by a time-symmetric method for ODEs and prove that it is always of an even order.

b. Determine which of the following methods are time symmetric:

1. The method $\mathbf{y}_{n+1} = \mathbf{y}_n + h\mathbf{f}(\mathbf{y}_n) + \frac{1}{2}h^2\mathbf{g}(\mathbf{y}_n)$, where $\mathbf{y}' = \mathbf{f}(y)$ and $\mathbf{y}'' = \mathbf{g}(\mathbf{y})$ (therefore $\mathbf{g}(\mathbf{y}) = [\partial\mathbf{f}(\mathbf{y})/\partial\mathbf{y}]\mathbf{f}(\mathbf{y})$);
2. The trapezoidal rule $\mathbf{y}_{n+1} = \mathbf{y}_n + \frac{1}{2}h[\mathbf{f}(\mathbf{y}_n) + \mathbf{f}(\mathbf{y}_{n+1})]$;
3. The Runge–Kutta method with the Butcher tableau

$$\begin{array}{c|c} \frac{1}{2} & \frac{1}{2} \\ \hline & 1 \end{array}.$$

SECTION B

6

Write an essay on multistep methods for ODEs, inclusive of their order, convergence and stability. Quote carefully all the theorems you are using and accompany your exposition with examples.

7

Write an essay on the Ritz method in finite element theory. Present complete proofs as appropriate and accompany your exposition with the example of the two-point boundary value problem $-(py')' + qy = f$, $x \in [-1, 1]$, with zero boundary conditions, given appropriate functions p, q, f.

END OF PAPER