UNIVERSITY OF CAMBRIDGE

MAMA/337, NST3AS/337, MAAS/337

MAT3
MATHEMATICAL TRIPOS Part III

Tuesday, 13 June, 2023 1:30 pm to 4:30 pm

PAPER 337

APPLICATIONS OF QUANTUM FIELD THEORY

Before you begin please read these instructions carefully

Candidates have TWO HOURS to complete the written examination.

Attempt **BOTH** questions.
There are **TWO** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Consider the $O(N)$ nonlinear sigma model in two space dimensions. The imaginary-time partition function can be written as

$$Z = \int \mathcal{D}\mathbf{n}\mathcal{D}\lambda e^{-S[\mathbf{n},\lambda]} ,$$

where $\mathbf{n}$ is an N-component vector and λ a Lagrange multiplier, such that

$$S[\mathbf{n},\lambda] = \frac{1}{2vg}\int d^2x \int_0^{1/T} d\tau \left[(\partial_\tau \mathbf{n})^2 + v^2(\nabla \mathbf{n})^2 + i\lambda(\mathbf{n}^2 - N)\right] .$$

Here T is the temperature, v a velocity and g a coupling.

(a) Perform the integral over $\mathbf{n}$ and thereby write the partition function as

$$Z = \int \mathcal{D}\lambda e^{-\widetilde{S}[\lambda]} , \tag{1}$$

for some $\widetilde{S}[\lambda]$ that you should express as a functional trace.

(b) Explain why at large N one can evaluate the partition function (1) using a saddle-point approximation. Write down the saddle point equation as a sum over Matsubara frequencies and integral over momenta, assuming the saddle-point value $i\lambda = m^2$ is independent of space and time.

(c) Write down the zero temperature limit of the saddle point equation as an integral. If necessary you may regulate any high energy or momentum divergences you find with a cutoff Λ, but take $\Lambda \to \infty$ where possible. Explain why a solution to the saddle point equation only exists for $g > g_c$, where g_c is a critical value of the coupling. Show that in this case m is given by

$$m = \frac{4\pi v(g - g_c)}{g g_c} . \tag{2}$$

Without solving any equations, state physically what will happen when $g < g_c$.

(d) Now consider $T > 0$, still with $g > g_c$. The solution to the saddle point equation will now depend on temperature, so we write $m(T)$. Denote the $T = 0$ solution (2) by $m = \Delta$, the zero temperature gap. The critical coupling g_c remains defined at $T = 0$. Recall that for reasonable functions $F(z)$:

$$T\sum_n F(i\omega_n) = \frac{-i}{4\pi}\int_C dz \coth\frac{z}{2T} F(z) , \tag{3}$$

where the contour C has two parts, one running down just to the left of the imaginary axis and one running up just to the right of the imaginary axis. Use the formula (3) to evaluate the sum over Matsubara frequencies in the $T > 0$ saddle point equation.

[QUESTION CONTINUES ON THE NEXT PAGE]

(e) Now perform the integral over momenta in the saddle point equation, using the expression you have just derived for the sum over Matsubara frequencies. Regularize your result with a cutoff Λ on the momenta. Remove the divergence by subtracting off the zero temperature saddle point equation at $g = g_c$, regularized in the same way, and then taking $\Lambda \to \infty$. In this way, obtain the result

$$m(T) = 2T \sinh^{-1}\left[\tfrac{1}{2} e^{\frac{\Delta}{2T}}\right] . \tag{4}$$

[*You may use without proof the integral*

$$\int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{1}{v^2k^2 + \omega^2} = \frac{1}{2kv} . \qquad]$$

(f) Obtain from (4) the leading temperature dependence of $m(T)$ at temperatures $T \ll \Delta$ and $T \gg \Delta$, and give a physical interpretation of your results.

(g) The Matsubara frequency Green's function of each component of the $\mathbf{n}$ field is

$$G(i\omega_n, k) = \frac{vg}{N} \frac{1}{v^2k^2 + \omega_n^2 + m(T)^2} .$$

Write down the corresponding retarded Green's function $G^R(\omega, k)$. Write down the spectral weight $\operatorname{Im} G^R(\omega, k)$. Explain how the retarded Green's function, using (4) to give $m(T)$, is consistent with quantum critical scaling.

2 The coherent state path integral of a spin-S antiferromagnetic Heisenberg model in two space dimensions, in a uniform background magnetic field $\mathbf{B}$, is given by

$$Z = \int \mathcal{D}\mathbf{n}_i \delta(|\mathbf{n}_i|^2 - 1) e^{iI[\{\mathbf{n}_i\}]} ,$$

where the action

$$I[\{\mathbf{n}_i\}] = S \sum_i \Gamma[\mathbf{n}_i] - \int dt \left[S^2 J \sum_{<ij>} \mathbf{n}_i \cdot \mathbf{n}_j + S \sum_i \mathbf{B} \cdot \mathbf{n}_i \right] . \qquad (\star)$$

Here the coupling $J > 0$, and the Wess-Zumino term $\Gamma[\mathbf{n}_i]$ is the oriented area of a region of the two-sphere target space of the $\mathbf{n}_i$ that is bounded by the curve $\mathbf{n}_i(t)$.

(a) Express $\mathbf{n}_i$ in terms of a slowly varying Néel order parameter $\widetilde{\mathbf{n}}(\mathbf{x}_i)$ plus a slowly varying fluctuation $\mathbf{m}(\mathbf{x}_i)$. The order parameter obeys $|\widetilde{\mathbf{n}}|^2 = 1$. The fluctuation is orthogonal to the order parameter, $\mathbf{m} \cdot \widetilde{\mathbf{n}} = 0$, and is small, $|\mathbf{m}| \ll |\widetilde{\mathbf{n}}|$.

(b) Argue that, in the continuum limit, the contribution to the Wess-Zumino term from the order parameter $\widetilde{\mathbf{n}}(\mathbf{x}_i)$ alone is negligible. You may ignore this contribution in the rest of this question.

(c) Given that the variation of the Wess-Zumino term obeys

$$S \int dt \delta\Gamma[\mathbf{n}] = S \int dt (\delta\mathbf{n}) \cdot \left(\mathbf{n} \times \frac{d\mathbf{n}}{dt} \right) ,$$

obtain the contribution to the continuum action from the Wess-Zumino term that is linear in $\mathbf{m}$.

(d) Obtain the continuum limit of the remainder of the action, i.e. coming from the terms in square brackets in $(\star)$. Assume that the model has been defined on a square lattice with lattice spacing a.

(e) Using the full continuum action you have obtained, integrate out the massive field $\mathbf{m}$ to obtain an effective action for the order parameter $\widetilde{\mathbf{n}}$.

[*You may quote any results you need concerning Gaussian integrals. Before performing the integral you will need to add a Lagrange multiplier term* $\lambda(\mathbf{x})\mathbf{m}(\mathbf{x}) \cdot \widetilde{\mathbf{n}}(\mathbf{x})$ *to the continuum Lagrangian, to impose the constraint* $\mathbf{m} \cdot \widetilde{\mathbf{n}} = 0$. *The effective action you obtain will be a function of* $\widetilde{\mathbf{n}}$ *and* λ.]

(f) Perform the Gaussian path integral over λ to obtain an effective action $I_{\text{eff}}[\widetilde{\mathbf{n}}]$.

(g) Write $\widetilde{\mathbf{n}}$ in terms of polar coordinates (θ, ϕ) on the two-sphere, and consider small fluctuations $\theta = \frac{\pi}{2} - \delta\theta$ and $\phi = \delta\phi$. Let the magnetic field point in the x direction. Show that the effective Lagrangian takes the form

$$\mathcal{L}_{\text{eff}} = -\alpha^2 \left((\nabla\delta\theta)^2 + (\nabla\delta\phi)^2 \right) + \beta^2 \left((\partial_t \delta\theta + B\delta\phi)^2 + (\partial_t \delta\phi - B\delta\theta)^2 \right) .$$

Here α and β are two real coefficients that you should give. By writing this Lagrangian in momentum/frequency space, or otherwise, obtain the dispersion relations of the linearized modes and interpret your result.

END OF PAPER