UNIVERSITY OF CAMBRIDGE

MAMA/336, NST3AS/336, MAAS/336

MAT3

MATHEMATICAL TRIPOS Part III

Thursday, 1 June, 2023 9:00 am to 11:00 am

PAPER 336

PERTURBATION METHODS

Before you begin please read these instructions carefully

Candidates have TWO HOURS to complete the written examination.

Attempt no more than **TWO** questions.
There are **THREE** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Consider functions $f_j(x_1, \ldots, x_m)$ of variables x_i, where $j = 1, \ldots, n$ and $i = 1, \ldots, m$, and their partial derivatives

$$f_{j_{x_i}} \equiv \frac{\partial f_j}{\partial x_i}, \quad j = 1, \ldots, n, i = 1, \ldots, m .$$

You are given that the Euler-Lagange equations for the Lagrangian $L\left(f_{j_{x_i}}, f_j\right)$, where $j = 1, \ldots, n$ and $i = 1, \ldots, m$, corresponding to the variational principle

$$\delta \int L\, d\mathbf{x} = 0 ,$$

are

$$\sum_{i=1}^{m} \frac{\partial}{\partial x_i} \left(\frac{\partial L}{\partial f_{j_{x_i}}} \right) - \frac{\partial L}{\partial f_j}, \quad j = 1, \ldots, n . \tag{1}$$

One dimensional waves can be described by a variational principle

$$\delta \int \int L(\phi_x, \phi_t, \phi) dx dt = 0 ,$$

where x and t are space and time variables, and $\phi(x, t)$ is the dependent variable. From equation (1), with $f_1 = \phi$, $x_1 = x$ and $x_2 = t$, the corresponding Euler-Lagrange equation for ϕ is

$$\frac{\partial L_1}{\partial x} + \frac{\partial L_2}{\partial t} - L_3 = 0 ,$$

where

$$L_1 = \frac{\partial L}{\partial \phi_x}, \quad L_2 = \frac{\partial L}{\partial \phi_t} \quad \text{and} \quad L_3 = \frac{\partial L}{\partial \phi} .$$

(a) Assume that the waves are modulated over length and times scales $X = \varepsilon x$ and $T = \varepsilon t$. Introduce a modulated phase function $\theta = \varepsilon^{-1}\Theta(X, T)$ such that

$$\phi(x, t) \equiv \Phi(\theta, X, T; \varepsilon) ,$$

where Φ is periodic in θ with normalised period 2π. Derive the exact equation

$$k\frac{\partial L_1}{\partial \theta} + \varepsilon\frac{\partial L_1}{\partial X} - \omega\frac{\partial L_2}{\partial \theta} + \varepsilon\frac{\partial L_2}{\partial T} - L_3 = 0 , \tag{2}$$

where

$$k(X, T) = \Theta_X , \quad \omega(X, T) = -\Theta_T ,$$

and

$$L \equiv L(k\Phi_\theta + \varepsilon\Phi_X, -\omega\Phi_\theta + \varepsilon\Phi_T, \Phi) .$$

[QUESTION CONTINUES ON THE NEXT PAGE]

Deduce that

$$\frac{\partial}{\partial \theta}\left(\left(kL_1 - \omega L_2\right)\Phi_\theta - L\right) + \varepsilon \frac{\partial}{\partial X}\left(\Phi_\theta L_1\right) + \varepsilon \frac{\partial}{\partial T}\left(\Phi_\theta L_2\right) = 0\,, \tag{3}$$

and thence that

$$\frac{\partial}{\partial X}\frac{\partial \overline{L}}{\partial k} - \frac{\partial}{\partial T}\frac{\partial \overline{L}}{\partial \omega} = 0\,, \tag{4}$$

where

$$\overline{L}(k, \omega, \Phi_\theta, \Phi_X, \Phi_T, \Phi; \varepsilon) = \frac{1}{2\pi}\int_0^{2\pi} L(k\Phi_\theta + \varepsilon\Phi_X, -\omega\Phi_\theta + \varepsilon\Phi_T, \Phi)\, d\theta\,.$$

Deduce the Euler-Lagrange equations, and their relationship to (2) and (4), for the variational principle

$$\delta \int\!\!\int\!\!\int L(k\Phi_\theta + \varepsilon\Phi_X, -\omega\Phi_\theta + \varepsilon\Phi_T, \Phi)\, d\theta dX dT = 0\,,$$

by viewing L as a function of $\Phi(\theta, X, T; \varepsilon)$, the modulated phase function $\Theta(X, T)$, and their derviatives.

(b) For the Lagrangian

$$L = \tfrac{1}{2}\phi_t^2 - \tfrac{1}{2}c^2(X,T)\phi_x^2\,,$$

deduce the linear equation satisfied by ϕ.

Henceforth assume that $0 < \varepsilon \ll 1$, and seek a leading-order solution of the form $\phi = A(X,T)\cos\theta(X,T)$.

(i) Derive the leading-order approximation of equation (3). By which name is this leading-order relation usually referred?

(ii) Expand $\overline{L}$ as

$$\overline{L} = \overline{L}^{(0)} + \varepsilon \overline{L}^{(1)} + \ldots\,.$$

Show that $\overline{L}^{(0)} \equiv \overline{L}^{(0)}(k, \omega, A)$, and thence deduce the Euler-Lagrange equation for variations of $\overline{L}^{(0)}$ with respect to A. Comment very briefly on your answer.

(iii) Deduce the conservation equation for wave action,

$$\frac{\partial W}{\partial T} + \frac{\partial}{\partial X}(c_g W) = 0\,,$$

where $W = \frac{\partial}{\partial \omega}\overline{L}^{(0)}$ and c_g (which is to be identified) is the group velocity.

2

(a) For $0 < \varepsilon \ll 1$ deduce the asymptotic behaviour of the integral,

$$I(\varepsilon) = \int_0^1 \frac{dx}{x(x+\varepsilon) + \varepsilon^3 \exp(-x^2)} \,,$$

up to and including terms of $O(1)$.

[*Hint: The following indefinite integrals may be quoted:*

$$\int \frac{dx}{x^2(x+1)^2} = \frac{2x^2-1}{x(x+1)} + 2\ln\left(1+\frac{1}{x}\right) \,,$$

$$\int \frac{x^2 dx}{(x+1)^2} = \frac{x(x+2)}{(x+1)} - 2\ln(1+x) \,. \qquad]$$

(b) By considering the largest terms in the series and using a discrete generalisation of Laplace's method, find the asymptotic value of the sum

$$S = \sum_{n=1}^{\infty} \frac{x^n}{(n-1)!n^n} \,,$$

for $x \gg 1$.

[*Hints: You may quote Stirling's formula, i.e. that*

$$m! \sim (2\pi m)^{\frac{1}{2}} \left(\frac{m}{e}\right)^m \quad \textit{for} \quad m \gg 1 \,,$$

and it may be helpful to recall that for a function, $f(t)$,

$$\int_a^b f(t)dt = \lim_{N\to\infty} \sum_1^N h f(t_n) \,,$$

where a and b are real, $h = (b-a)/N$ is the interval length, and t_n is any point in the interval $[a+(n-1)h, a+nh]$.]

3 For $0 \leqslant x < \infty$, the function $y(x;\varepsilon)$ satisfies the differential equation

$$\varepsilon x \frac{d^2 y}{dx^2} + (x + 2\,\varepsilon\,\mathrm{sech}\,x)\,\frac{dy}{dx} + \left(\varepsilon x^3 + x + 1\right) y = 0\,,$$

where $0 < \varepsilon \ll 1$, together with the boundary condition that

$$y(0;\varepsilon) = 1\,.$$

Find the leading-order matched asymptotic solution for $y(x;\varepsilon)$ for $0 \leqslant x < \infty$. Clearly delineate the three asymptotic regions that you identify, and explain why the single boundary condition given is sufficient to specify a unique solution.

[*Hint: The governing equation in one of the regions has a first integral.*]

END OF PAPER