UNIVERSITY OF CAMBRIDGE

MAMA/335, NST3AS/335, MAAS/335

MAT3

MATHEMATICAL TRIPOS Part III

Thursday, 8 June, 2023 1:30 pm to 3:30 pm

PAPER 335

DIRECT AND INVERSE SCATTERING OF WAVES

Before you begin please read these instructions carefully

Candidates have TWO HOURS to complete the written examination.

Attempt no more than **TWO** questions.
There are **THREE** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Consider a time-harmonic plane wave in the plane (x, z) (time dependence $e^{-i\omega t}$), incident from $z > 0$ upon the interface $z = 0$ between two homogeneous half-spaces, with wave speed c_0 in the upper medium ($z > 0$) and c_1 in the lower medium ($z < 0$), and let $n = c_0/c_1$ be the index of refraction of the lower medium with respect to the upper medium.

(i) Given boundary conditions at $z = 0$

$$\begin{aligned} \psi_0 &= \psi_1 \\ \frac{\partial \psi_0}{\partial z} &= \frac{\partial \psi_1}{\partial z} \ , \end{aligned}$$

and using the cosine form of Snell's law:

$$n \cos\theta_T = (1+\alpha)^{1/2} \cos\theta_i \quad , \quad \text{where} \quad \alpha = \frac{n^2 - 1}{\cos^2\theta_i} \qquad (1)$$

which relates the cosines of the incident and transmitted angles θ_i and θ_T, derive first the reflected and the transmitted field for the *exact solution*.

(ii) Noting that because of the geometry of this problem we can write the total field as $\psi(x, z) = f(z)e^{ik\sin\theta_i x}$, derive the approximate reflected field for

(a) the first Born approximation

(b) the first Rytov approximation

[Hint: use the result from the first Born approximation to write the first Rytov approximation.]

(iii) Assuming that

$$\alpha \ll 1 \quad ,$$

it is possible to expand $n\cos\theta_T$ as given by equation (1) above. Using the parameter α, compare the first order Born and Rytov approximations for the reflected field, and relate them to the exact solution.

2 Consider the two-dimensional problem of propagation of a time-harmonic plane wave initially given by $\psi(x,z) = e^{ikx}$.

(i) Assume the wave is propagating in free space and incident at $x = -\xi$ onto a vertical layer in the region $x \in [-\xi, 0]$. The layer has refractive index $n(z) = 1 + \mu W(z)$, where $\mu^2 \ll 1$, and $W(z)$ is a continuous random fluctuation with mean zero, stationary in z, and normally distributed. We assume that the field acquires only a phase change on going through the layer, which therefore acts as a 'phase screen'.

By considering the parabolic equation for the reduced field $E(x,z) = \psi e^{-ikx}$, and using a Taylor expansion of $E(x,z)$ about $x = 0$, show that the field will acquire amplitude fluctuations after travelling a short distance in free space beyond the phase screen, even though the screen only imposes a phase change.

(ii) Now consider an extended, two-dimensional, weakly scattering random medium with refractive index $n(x,z) = 1 + \mu W(x,z)$, and μ as before and $W(x,z)$ again with mean zero, stationary in x and z, and normally distributed (hence the field obeys the parabolic equation).

Derive an approximate equation for the fourth moment of the field m_4 defined by

$$m_4 = < E(x,z_1)E^*(x,z_2)E(x,z_3)E^*(x,z_4) > ,$$

which should include terms dependent on the autocorrelation function of the medium $\rho(\xi, \zeta_{ij}) = < W(x,z_i)W(x',z_j) >$. State carefully any assumptions made in the derivation.

(iii) The equation for the fourth order moment can be simplified by using the new variables

$$\begin{aligned} R &= \frac{1}{4}(z_1 + z_2 + z_3 + z_4), \qquad \rho = (z_1 - z_2 + z_3 - z_4) \\ r_1 &= \frac{1}{2}(z_1 + z_2 - z_3 - z_4), \qquad r_2 = \frac{1}{2}(z_1 - z_2 - z_3 + z_4) \ , \end{aligned}$$

which allow the wave to be written as a function of r_1 and r_2 only, so the parabolic equation for m_4 becomes

$$\frac{\partial m_4}{\partial x} = \frac{i}{k}\left(\frac{\partial^2}{\partial r_1 \partial r_2} - Q\right) m_4 \ , \qquad (1)$$

where Q denotes the term accounting for the scattering effects of the medium and is independent of x.

Consider again propagation through a thin screen as in (i), with thickness $\Delta x \ll 1$.
(a) By writing m_4 as $m_4 = \exp(\Psi)$, and using (1) to model propagation within the thin screen, derive m_4 at the exit from the screen at $x = 0$ (discard any terms of order $(\Delta x)^2$ or higher).

(b) Considering now the propagation in free space beyond the thin screen, derive an expression for the fourth moment m_4 at $x = L$.
[Hint: you may wish to take Fourier transform w.r.t. to r_1 and r_2.]

3

(i) Consider the inverse problem defined as finding x from

$$Ax = y^{(\delta)} \quad , \tag{1}$$

where A is a symmetric positive definite $n \times n$ matrix, and $y^{(\delta)}$ is given data with known error bound δ w.r.t. to the exact data y, such that $\| y^{(\delta)} - y \| \leqslant \delta$.

By using Singular Value Decomposition for A, relate the stability of this inverse problem to the value of the ratio between the largest and the smallest eigenvalues of A, $\kappa = \lambda_n / \lambda_1$.

(ii) Let $A : X \mapsto Y$ be a compact operator between Hilbert spaces X and Y.
Define the Moore-Penrose generalised inverse of the operator A.
Hence, define a regularisation strategy R_α for the inverse problem of finding f from $Af = g^{(\delta)}$ and show that Tikhonov regularisation is a regularisation strategy for this, with an appropriate choice of α.

(iii) Now consider the operator $A : \mathrm{L}^2([0,1]) \mapsto \mathrm{L}^2([0,1])$, defined for all $f(x)$ in $\mathrm{L}^2([0,1])$ by

$$Af(x) = \int_0^x f(x')dx' \ . \tag{2}$$

Check that $\{\sigma_n; u_n; v_n\}$, where

$$\sigma_n = \frac{2}{(2n-1)\pi} \quad , \quad u_n(x) = \sqrt{2} \cos \frac{x}{\sigma_n} \quad , \quad v_n(x) = \sqrt{2} \sin \frac{x}{\sigma_n} \tag{3}$$

is a singular value system for A.

Hence use Tikhonov regularisation to write explicitly the regularised solution f_α for this problem.

END OF PAPER