UNIVERSITY OF CAMBRIDGE

MAMA/333, NST3AS/333, MAAS/333

MAT3

MATHEMATICAL TRIPOS Part III

Thursday, 1 June, 2023 1:30 pm to 4:30 pm

PAPER 333

FLUID DYNAMICS OF CLIMATE

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

Attempt no more than **THREE** questions.
There are **FOUR** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

The shallow water potential vorticity is $q = (f + \zeta)/h$, where $f = f_0 + \beta y$ is the Coriolis parameter, $\zeta = (\nabla \times \mathbf{u}) \cdot \hat{\mathbf{z}}$ is the vertical component of the relative vorticity, $h(x, y, t) = H_0 - h_b(x, y) + \eta(x, y, t)$ is the fluid depth, H_0 is the mean fluid depth, h_b is the bottom height, and η is the surface height measured relative to the surface height for a fluid at rest. Show, under a set of assumptions which you should clearly state, that q can be approximated by the shallow water quasi-geostrophic potential vorticity, P_g, where

$$P_g = \frac{f_0}{H_0}\left(1 + \frac{\beta y}{f_0} + \frac{\nabla^2\psi}{f_0} + \frac{h_b}{H_0} - \frac{f_0\psi}{gH_0}\right), \tag{1}$$

and $\psi(x, y, t)$ is the quasi-geostrophic streamfunction. Provide a physical interpretation for the terms in P_g involving ψ.

Consider a situation where the bottom height is a linear function of y, i.e. $h_b = \alpha y$, where $\alpha y \ll H_0$. Derive the dispersion relation for small amplitude quasi-geostrophic perturbations to a basic state where the fluid is at rest. Using the principle of conservation of potential vorticity and with use of a diagram, discuss the influence of β and α on the direction of phase propagation of the waves.

Now, consider a situation where $\eta = 0$ (the rigid lid approximation). Starting from shallow water potential vorticity conservation, $Dq/Dt = 0$, and assuming that the Rossby number is small, but *without* making the assumption that h_b is small compared to the mean depth, obtain an equation for the evolution of the streamfunction. If the fluid depth is an exponential function of y, i.e. $h(y) = H_0 \exp(\gamma y)$, derive the dispersion relation for small amplitude perturbations to a state of rest. Discuss the relative importance of β and γ in this case and identify a criterion for the influence of β or γ to be the dominant effect.

2 The incompressible, inviscid, hydrostatic, Boussinesq equations can be written

$$\frac{\partial \mathbf{u}_h}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u}_h + f\hat{\mathbf{z}} \times \mathbf{u}_h = -\frac{1}{\rho_0}\nabla_h p, \tag{1}$$

$$\frac{1}{\rho_0}\frac{\partial p}{\partial z} = b, \tag{2}$$

$$\frac{\partial b}{\partial t} + \mathbf{u} \cdot \nabla b = 0, \tag{3}$$

$$\nabla \cdot \mathbf{u} = 0, \tag{4}$$

where $b = -g\rho/\rho_0$ is the buoyancy, ρ is the fluid density measured relative to a reference density ρ_0, $\mathbf{u} = (u, v, w)$ is the velocity vector, $\mathbf{u}_h = (u, v)$ is the horizontal velocity vector, $\nabla_h = (\partial_x, \partial_y)$, $\hat{\mathbf{z}}$ is the unit vector in the local vertical direction, and f is the Coriolis parameter which may be considered constant.

Consider a basic state with buoyancy $b = M^2 x + N^2 z$, and a velocity in thermal wind balance. Derive the dispersion relation for small amplitude, y-independent perturbations to the basic state in an unbounded domain where k and m are the wavenumbers in the x and z directions, respectively. Clearly state any additional assumptions. Show that solutions are oscillatory in time if $M^4 < N^2 f^2$.

For the case that $M^4 < N^2 f^2$, find the orientation of lines of constant phase in the x-z plane for perturbations oscillating at the minimum possible frequency. Compare the orientation of the minimum frequency modes with the orientation of surfaces of constant buoyancy. Discuss the restoring force(s) and compare the minimum frequency with the frequency of oscillations in an unstratified, rotating fluid.

For modes with a given frequency, ω, find an expression for θ, the angle of the wavenumber vector (k, m) as measured counter-clockwise from the horizontal direction $(\hat{\mathbf{x}})$.

Consider waves that are generated by a distant source with a frequency $\omega^2 > f^2$ and with a group velocity vector, $\mathbf{c}_g$, that points down and to the right, such that $\mathbf{c}_g \cdot \hat{\mathbf{z}} < 0$ and $\mathbf{c}_g \cdot \hat{\mathbf{x}} > 0$. These waves then reflect off a flat, rigid horizontal boundary. Find an expression for θ (as defined above) for the reflected waves. Sketch phase lines of the incident and reflected waves and surfaces of constant buoyancy. On your sketch, label the group velocity for the incident and reflected waves.

3

The f-plane quasi-geostrophic equations for a fluid with constant buoyancy frequency N confined between two rigid boundaries at $z = 0$ and $z = H$ are:

$$\{\frac{\partial}{\partial t} + \mathbf{u}_g.\nabla\}(\psi_{xx} + \psi_{yy} + \frac{f^2}{N^2}\psi_{zz}) = 0 \quad \text{in } 0 < z < H \tag{1}$$

$$\{\frac{\partial}{\partial t} + \mathbf{u}_g.\nabla\}\psi_z = 0 \quad \text{on } z = 0 \tag{2}$$

$$\{\frac{\partial}{\partial t} + \mathbf{u}_g.\nabla\}\psi_z = 0 \quad \text{on } z = H, \tag{3}$$

where ψ is the quasi-geostrophic streamfunction and the geostrophic velocity $\mathbf{u}_g$ has components $(-\psi_y, \psi_x, 0)$.

Give a brief description of the physical meaning of each of equations (1), (2) and (3).

Now consider small-amplitude disturbances to a background flow $(\Lambda z, 0, 0)$. Carefully derive linearised forms of (1), (2) and (3), describing the evolution of $\psi'(x, y, z, t)$, the disturbance part of the quasi-geostrophic streamfunction.

Add a term $-\alpha\psi'_z$, with $\alpha > 0$ and constant, to the right-hand side of the linearised form of (2), to represent the damping effect of physical processes acting near the lower boundary.

Now seek solutions of the linearised equations, with the α term included, of the form $\psi'(x, y, z, t) = \mathrm{Re}(\hat{\psi}(z, t)e^{ikx})$. You may assume that $\psi'_{xx} + \psi'_{yy} + (f^2/N^2)\psi'_{zz}$ is zero in $0 < z < H$.

Show that under this assumption, for given k there are constants $A(k)$ and $B(k)$ such that

$$\hat{\psi}(0, t) = -A(k)\hat{\psi}_z(0, t) + B(k)\hat{\psi}_z(H, t), \tag{4}$$

$$\hat{\psi}(H, t) = -B(k)\hat{\psi}_z(0, t) + A(k)\hat{\psi}_z(H, t), \tag{5}$$

where, defining $\mu = NHk/f$,

$$A(k) = (f/Nk)\coth\mu \text{ and } B(k) = (f/Nk)\mathrm{cosech}\,\mu.$$

[Hint: write $\hat{\psi}(y) = C\cosh(kNz/f) + D\cosh(kN(H - z)/f)$.]

Derive a pair of coupled ordinary differential equations in time for the quantities $\hat{\psi}_z(0, t)$ and $\hat{\psi}_z(H, t)$.

[QUESTION CONTINUES ON THE NEXT PAGE]

Now assume that $\hat{\psi}(z,t) = \hat{\Psi}(z)e^{-ikct}$. Define the quantities $\tilde{c} = c/(\Lambda H)$ and $\epsilon = \alpha/(k\Lambda H)$ and deduce the dispersion relation

$$\tilde{c} = \tilde{c}_{\pm}(\mu,\epsilon) = \tfrac{1}{2}(1 - i\epsilon) \pm \sqrt{\left(\frac{1}{\mu^2} - \frac{\coth\mu}{\mu} + \frac{1}{4}\right) + i\epsilon\left(\frac{1}{2} - \frac{\coth\mu}{\mu}\right) - \frac{\epsilon^2}{4}}. \qquad (6)$$

Consider first the case $\epsilon = 0$.

(i) Explain why (6) implies instability for $\mu < \mu_c$ and stability for $\mu > \mu_c$ where an equation for μ_c should be given. Find approximations for $\tilde{c}_{\pm}$ for μ large and give a physical interpretation.

Now consider $\epsilon > 0$.

(ii) Show that for $\mu = \mu_c$ there is instability.

(iii) Analyse and interpret the behaviour of $\tilde{c}_{\pm}$ for μ large. (Consider terms up to and including $O(\mu^{-1})$.)

4 (a) Consider the linearised Boussinesq primitive equations on an equatorial β-plane, assuming that geostrophic balance holds in the y-momentum equation, written in the form

$$u_t - \beta y v = -\phi_x, \tag{1}$$
$$\beta y u = -\phi_y, \tag{2}$$
$$\sigma = \phi_z, \tag{3}$$
$$\sigma_t + N^2 w = 0, \tag{4}$$
$$u_x + v_y + w_z = 0, \tag{5}$$

where N is the constant buoyancy frequency and ϕ and σ represent rescaled pressure and density perturbations.

The corresponding form of the shallow-water equations is

$$U_t - \beta y V = -\Phi_x, \tag{6}$$
$$\beta y U = -\Phi_y, \tag{7}$$
$$\Phi_t + c^2(U_x + V_y) = 0. \tag{8}$$

Show that if $U(x,y,t)$, $V(x,y,t)$ and $\Phi(x,y,t)$ are solutions of (6)-(8), then $u(x,y,z,t) = \chi(z)U(x,y,t)$, $v(x,y,z,t) = V(x,y,t)\chi(z)$, $\phi(x,y,z,t) = \Phi(x,y,t)\chi(z)$, are solutions of (1)-(5) provided that $\chi(z)$ satisfies a second-order differential equation.

Give the explicit form of the equation for $\chi(z)$. Give also the corresponding expressions for $w(x,y,z,t)$ and $\sigma(x,y,z,t)$ in terms of U, V, Φ and χ.

(b) Now consider equatorially trapped plane-wave solutions of (6)-(8), of the form $U = \mathrm{Re}(\hat{U}(y)e^{i(kx-\omega t)})$, $V = \mathrm{Re}(\hat{V}(y)e^{i(kx-\omega t)})$ and $\Phi = \mathrm{Re}(\hat{\Phi}(y)e^{i(kx-\omega t)})$. Show that there is a Kelvin wave, with $V = 0$, with dispersion relation $\omega = kc$ and derive the corresponding forms of $\hat{U}$ and $\hat{\Phi}$. By expressing $\hat{U}(y)$ and $\hat{\Phi}(y)$ in terms of $\hat{V}(y)$ and then using the information on the eigenvalue problem given below at the end of the question, show that there is a family, labelled by $n = 1, 2, 3, \ldots$ of Rossby waves with dispersion relation $\omega = -kc/(2n+1)$. Give the corresponding forms of $\hat{U}$, $\hat{V}$ and $\hat{\Phi}$ for $n = 1$.

Draw a sketch in the k, ω plane of the dispersion relations for different wave modes that are found from the full equatorial β-plane shallow-water equations, without an assumption of geostrophic balance in the y-momentum equation, and comment briefly on the differences between these dispersion relations and those found above.

(c) Use your results from (a) and (b) to deduce the properties of equatorially trapped waves in the Boussinesq system (1)-(5) described by solutions of the form $u(x,y,z,t) = \mathrm{Re}(\hat{u}(y)e^{i(kx+mz-\omega t)})$, with corresponding forms for other variables. What are the expressions for ω in terms of k and m?

Consider a fluid occupying the region $z > 0$. A forcing with frequency $\omega = N\sin\theta$, with $\theta \ll 1$, is applied at the boundary $z = 0$ in a localized region in x. Assume that the Kelvin wave and the $n = 1$ Rossby wave are excited. Draw a diagram of the (x,z) plane showing the regions where these different wave responses will appear. Within these regions, what direction of phase propagation will be observed?

[You may assume that the eigenvalue problem $\psi_{yy} - y^2\psi = \lambda\psi$ with $|\psi| \to 0$ as $|y| \to \infty$ has eigenvalues $\lambda_n = -(2n+1)$ for $n = 0, 1, \ldots$ with corresponding eigenfunctions $\psi_n(y) = H_n(y)\exp(-y^2/2)$, where the $H_n(.)$ are the Hermite polynomials, with $H_0(s) = 1$, $H_1(s) = 2s$, $H_2(s) = 4s^2 - 2$, etc.]

END OF PAPER