UNIVERSITY OF CAMBRIDGE

MAMA/331, NST3AS/331, MAAS/331

MAT3
MATHEMATICAL TRIPOS Part III

Monday, 12 June, 2023 9:00 am to 11:00 am

PAPER 331

HYDRODYNAMIC STABILITY

Before you begin please read these instructions carefully

Candidates have TWO HOURS to complete the written examination.

Attempt no more than **TWO** questions.
There are **THREE** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

The Rayleigh equation for examining the stability of an inviscid flow is

$$\phi'' - \alpha^2\phi - \frac{U''}{U-c}\phi = 0$$

where $\psi(x,z,t) = \phi(z)e^{i\alpha(x-ct)}$ is the streamfunction of the disturbance, $U(z)$ is the base profile, $'$ indicates derivative with respect to z, α is the wavenumber and c is the complex wave speed.

(i) Assuming walls at $z = 0$ and $z = 1$, prove

(a) the inflexion point criterion for instability.

(b) Howard's semicircle theorem.

(ii) When $U(z) = 1 - |z|$ for $|z| < 1$ and $U(z) = 0$ for $|z| > 1$, show that c for a symmetric mode where $\phi(z) = \phi(-z)$ satisfies

$$2\alpha^2c^2 + \alpha(1 - 2\alpha - e^{-2\alpha})c - (1 - \alpha - (1+\alpha)e^{-2\alpha}) = 0.$$

Hence demonstrate that this mode is unstable for $1 \leqslant \alpha < \alpha_s < 2$ where the threshold α_s need not be computed.

[Hint: $e^{-4} \approx \frac{1}{55}$]

2

(a) Consider two fixed vectors $\mathbf{v}$ and $\mathbf{w}$ where $|\mathbf{v}|^2 = c_1^2$, $|\mathbf{w}|^2 = c_2^2$ and $\mathbf{v} \cdot \mathbf{w} = \alpha c_1 c_2$ ($|\alpha| \leqslant 1$). If $\mathbf{u}(t) = \mathbf{v}e^{-\lambda_1 t} + \mathbf{w}e^{-\lambda_2 t}$ with $\lambda_1, \lambda_2 > 0$, find the condition on α for there to be initial energy growth for some c_1 and c_2 when energy is defined as $|\mathbf{u}|^2$. If $c_2 = 1$, find the optimal ratio $\mu := c_1/c_2$ for maximum initial energy growth as a function of α.

(b) You are given that the dimensionless nonlinear equations governing perturbations of the basic state of a conducting motionless state ($\mathbf{U} = \mathbf{0}$ and $\Theta = -z$) in Rayleigh-Bénard convection are

$$\begin{aligned} \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} &= -\nabla p + \sigma Ra\theta \hat{\mathbf{z}} + \sigma \nabla^2 \mathbf{u}, \\ \nabla \cdot \mathbf{u} &= 0, \\ \frac{\partial \theta}{\partial t} - w + \mathbf{u} \cdot \nabla \theta &= \nabla^2 \theta \end{aligned}$$

where Ra is the Rayleigh number which is a non-dimensional measure of the temperature difference between the two boundaries at $z = 0$ and $z = 1$. The boundary conditions are stress-free on the velocity field and $\theta = 0$ at $z = 0, 1$ and periodicity is assumed across the other boundaries of the domain $V := \{(x, y, z) \in [0, L]^2 \times [0, 1]\}$.

(i) Show that a composite energy $E := \frac{1}{2} \int_V \mathbf{u}^2 + \sigma Ra\theta^2 dV$ evolves as follows

$$\frac{dE}{dt} = \sigma \int_V \left[2Ra\, w\theta - |\nabla \mathbf{u}|^2 - Ra|\nabla \theta|^2 \right] dV \qquad (*)$$

where $w = \mathbf{u} \cdot \hat{\mathbf{z}}$. What happens to this energy if $Ra = 0$?

(ii) Using the constraint that $\nabla \cdot \mathbf{u} = 0$ with the Lagrange multiplier field $2p(\mathbf{x})$, show that the variational principle to stationarize the integral on the right hand side of $(*)$ gives the Euler-Lagrange equations

$$\begin{aligned} 0 &= -\nabla p + \sigma Ra\theta \hat{\mathbf{z}} + \sigma \nabla^2 \mathbf{u}, \\ 0 &= w + \nabla^2 \theta, \\ \nabla \cdot \mathbf{u} &= 0. \end{aligned}$$

Confirm that if these equations have a solution, then the energy in $(*)$ does not initially decay.

(iii) Now relate the linear eigenvalue problem assuming $(u, \theta, p) \propto e^{\lambda t}$ to the problem in (ii). If you are told that the eigenvalues are all real and the largest eigenvalue λ_m of the linear stability problem first reaches 0 (from below) as Ra reaches Ra_{crit} (from below), deduce what this means for the energy growth possible on the basic state.

(iv) Confirm that the linear operator $\mathcal{L}$ you have written down in (iii) is normal by showing that it is self-adjoint under the inner product implied in $(*)$ i.e.

$$\langle \Phi_i, \Phi_j \rangle := \int_V \left[\mathbf{u}_i \mathbf{u}_j + \sigma Ra\, \theta_i \theta_j \right] dV$$

where $\Phi_i = (\mathbf{u}_i, \theta_i, p_i)$ and $\Phi_j = (\mathbf{u}_j, \theta_j, p_j)$.

3

Consider the stability of an inviscid Boussinesq fluid in a frame rotating with steady angular velocity $\Omega\hat{\mathbf{z}}$ such that the equations are

$$\frac{\partial \mathbf{u}^*}{\partial t^*} + \mathbf{u}^* \cdot \nabla^* \mathbf{u}^* + 2\Omega\,\hat{\mathbf{z}} \times \mathbf{u}^* = -\frac{1}{\rho_0}\nabla^* p^* + \alpha g(\theta^* - \theta_0^*)\hat{\mathbf{z}},$$
$$\frac{\partial \theta^*}{\partial t^*} + \mathbf{u}^* \cdot \nabla^* \theta^* = 0,$$
$$\nabla^* \cdot \mathbf{u}^* = 0$$

where starred variables have dimensions, with boundary conditions that $\mathbf{u}^* \cdot \hat{\mathbf{y}} = 0$ on $y = 0$ and L, and $\mathbf{u}^* \cdot \hat{\mathbf{z}} = 0$ on $z = 0$ and L ($\mathbf{g} = g\hat{\mathbf{z}}$ is the acceleration due to gravity and α is the coefficient of thermal expansion).

(a) Introducing non-dimensional variables as follows

$$\mathbf{u} := \mathbf{u}^*/V, \quad \theta := (\theta^* - \theta_0^*)/\Delta\theta, \quad p := p^*/(2\rho_0\Omega L V), \quad t := Vt^*/L, \quad \mathbf{x} := \frac{\mathbf{x}^*}{L},$$

show how the equations non-dimensionalise to

$$Ro\left[\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u}\right] + \hat{\mathbf{z}} \times \mathbf{u} = -\nabla p + \frac{B}{Ro}\theta\hat{\mathbf{z}},$$
$$\frac{\partial \theta}{\partial t} + \mathbf{u}\cdot\nabla\theta = 0,$$
$$\nabla \cdot \mathbf{u} = 0$$

where

$$Ro := \frac{V}{2\Omega L}, \qquad B := \frac{\alpha g \Delta\theta}{4\Omega^2 L}$$

are respectively the Rossby and Burgers numbers.

(b) Confirm that

$$\mathbf{U} = z\hat{\mathbf{x}}, \quad \Theta = z - \frac{Ro}{B}y$$

can form a steady basic state with an appropriate pressure field P which should be determined.

(c) Linearise the equations in (a) around the base state of (b) using the expansions

$$\mathbf{u} = (z + u')\hat{\mathbf{x}} + v'\hat{\mathbf{y}} + Ro\, w'\hat{\mathbf{z}},$$
$$p = P + p',$$
$$\theta = \Theta + Ro\,\theta'.$$

(d) Using your linearised equations in (c), show that the vertical perturbation vorticity equation is

$$\left(\frac{\partial}{\partial t} + z\frac{\partial}{\partial x}\right)\left(\frac{\partial v'}{\partial x} - \frac{\partial u'}{\partial y}\right) = \frac{\partial w'}{\partial z} + Ro\,\frac{\partial w'}{\partial y}.$$

[QUESTION CONTINUES ON THE NEXT PAGE]

(e) By considering the $Ro \to 0$ limit of the u', v' and w' equations, show that the linear problem can be reduced to

$$\left(\frac{\partial}{\partial t} + z\frac{\partial}{\partial x}\right)\left(\frac{\partial^2 p'}{\partial z^2} + B\left[\frac{\partial^2 p'}{\partial x^2} + \frac{\partial^2 p'}{\partial y^2}\right]\right) = 0$$

with boundary conditions

$$\left.\frac{\partial p'}{\partial x} = 0\right|_{y=0,1} \quad \& \quad \left.\left(\frac{\partial}{\partial t} + z\frac{\partial}{\partial x}\right)\frac{\partial p'}{\partial z} = \frac{\partial p'}{\partial x}\right|_{z=0,1}.$$

(f) Look for normal mode solutions of your problem in (e) of the form

$$p' = p(z)\sin(n\pi y)e^{i\alpha(x-ct)}$$

and hence show

$$c = \frac{1}{2} \pm \frac{\sqrt{(\lambda\coth\lambda - 1)(\lambda\tanh\lambda - 1)}}{2\lambda}$$

where $\lambda := \frac{1}{2}\sqrt{B(\alpha^2 + n^2\pi^2)}$ for $n = 1, 2, 3, \ldots$.

(g) Deduce that instability only occurs when $\lambda\tanh\lambda < 1$.

END OF PAPER