MAMA/329, NST3AS/329, MAAS/329

MAT3

MATHEMATICAL TRIPOS Part III

Monday, 5 June, 2023 1:30 pm to 4:30 pm

PAPER 329

SLOW VISCOUS FLOW

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

Attempt **ALL** questions.
There are **THREE** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 (a) State the Papkovich–Neuber representation for the velocity and pressure in Stokes flow. Use this representation, explaining your choice of trial harmonic potentials, to determine the velocity field due to a rigid sphere of radius a moving with velocity $\mathbf{U}$ through unbounded fluid of viscosity μ.

(b) Two rigid spheres of radius a are a vector distance $\mathbf{R}$ apart in unbounded fluid, where $a \ll R$. The first sphere is acted on by a constant force $\mathbf{F} = 6\pi\mu a\mathbf{V}$, the second sphere is force free, and both spheres are couple free.

Find the velocity $\mathbf{U}_2$ of the second sphere, correct to $O(Va^3/R^3)$, assuming that the velocity due to the first sphere is unperturbed to this order.

[*You may assume the Faxén formula* $\mathbf{U} = \dfrac{\mathbf{F}}{6\pi\mu a} + \mathbf{u}_\infty + \dfrac{a^2}{6}\nabla^2\mathbf{u}_\infty$, *but should explain how you apply it.*]

Use scaling arguments to explain why the velocity $\mathbf{U}_1$ of the first sphere differs from $\mathbf{V}$ by $O(Va^4/R^4)$. Explain why the next correction to $\mathbf{U}_2$ is not $O(Va^5/R^5)$, but $O(Va^7/R^7)$.

(c) Cartesian coordinates are defined for the problem in part (b) such that $\mathbf{V} = (V, 0, 0)$, with $V > 0$, and that the centres of the two spheres are at $\big(X_1(t), Y_1(t), 0\big)$ and $\big(X_2(t), Y_2(t), 0\big)$, respectively. At $t = 0$, $X_1 = X_2 = Y_1 = 0$ and $Y_2 = Y_0 \gg a$.

Using the results in part (b), and explaining any further approximations, show that

$$\frac{\mathrm{d}Y_2/\mathrm{d}t}{\mathrm{d}X_1/\mathrm{d}t} = -\frac{3a}{4}\frac{X_1 Y_0}{(X_1^2 + Y_0^2)^{3/2}} \,.$$

Deduce the leading-order approximation to $\lim_{t\to\infty} Y_2(t) - Y_0$. What happens to $X_2(t)$ as $t \to \infty$?

2 A planar sheet of fluid of viscosity μ undergoes extension. With respect to Cartesian axes, the sheet occupies $-h(x,t) \leqslant z \leqslant h(x,t)$. There is no flow or variation in the y-direction, so that the velocity $\mathbf{u}(x,z,t) = (u,0,w)$. The sheet is acted upon by surface tension, with constant coefficient γ, but the effects of gravity and inertia are negligible.

(a) Assuming that $\partial h/\partial x \ll 1$, explain why u is approximately independent of z and derive equations for $w(x,z,t)$ and σ_{zz}. Deduce that

$$\sigma_{xx} = -p_a + \gamma\frac{\partial^2 h}{\partial x^2} + 4\mu\frac{\partial u}{\partial x},$$

where p_a is the uniform pressure outside the sheet.

Draw a diagram to show all of the forces acting on a fluid slice of length δx and varying thickness. Deduce that

$$\frac{\partial}{\partial x}\left(4\mu h\frac{\partial u}{\partial x}\right) + \gamma h\frac{\partial^3 h}{\partial x^3} = 0\,. \tag{1}$$

Obtain a second relationship between $h(x,t)$ and $u(x,t)$ using mass conservation.

(b) Two cylindrical gas bubbles of radius a are gently pressed against one another by a weak external flow in the surrounding liquid. Deformation of the bubbles is negligible except in a flat region of fixed length $2L$ where the bubbles are separated by a thin liquid sheet of thickness $h_0(x,t) \ll L$.

Making reference to the pressure in the flat region and in the external fluid, explain why the liquid drains out of the sheet.

Within the flat region, the sheet thickness h_0 is initially independent of x. Using (1), show that $u = Ux/L$ for some $U(t)$. Deduce that h_0 remains independent of x and obtain an ordinary differential equation for $h_0(t)$ in terms of U.

(c) The value of U is controlled by short transition regions at each end of the sheet over which the interfacial curvature changes from 0 to $1/a$. The lengthscale δ of these regions satisfies $h_0 \ll \delta \ll L$. Use scaling arguments to show that (i) $\delta \sim (ah_0)^{1/2}$, (ii) $U \sim (\gamma/\mu)(h_0/a)^{1/2}$ and (iii) the flux uh is approximately constant throughout the transition region.

(d) Use scaled variables $\xi \equiv (x-L)/(ah_0)^{1/2}$ and $H = h/h_0$ to rescale (1) in the transition region, and eliminate u to obtain a third-order differential equation for $H(\xi)$. Integrate this twice to show that

$$H^{-1/2}H_\xi = \frac{8V}{3}\left(1 - H^{-3/2}\right)\,,$$

where V is a suitable dimensionless parameter. [*Hint*: The second integration uses an integrating factor.] By considering the behaviour of H as $\xi \to \infty$, show that

$$U = \frac{3\gamma}{8\mu}\left(\frac{2h_0}{a}\right)^{1/2}.$$

(e) Hence determine $h_0(t)$ and comment on its large-time behaviour.

3 A thin layer of viscous fluid flows steadily down a rigid plane that is inclined at angle $\alpha \ll 1$ to the horizontal. Far upslope the layer has uniform thickness h_0, but further down the slope something causes variations in thickness. Surface tension is negligible.

(a) Use the equations of lubrication theory to derive the dimensionless equation

$$\boldsymbol{\nabla}\cdot(h^3\mathbf{e}_x) = \boldsymbol{\nabla}\cdot(h^3\boldsymbol{\nabla}h)\ ,$$

where $\mathbf{e}_x$ is a unit vector pointing downslope, $h(x,y) \to 1$ as $x \to -\infty$ and the dimensionless variables should be defined.

(b) Suppose first there is no cross-slope variation. Show that $h(x) \sim 1 + Ae^{kx}$ as $x \to -\infty$, where A and k are constants and k is to be determined. If $A > 0$ show that

$$h \sim x - x_0 + O(x^{-2}) \text{ as } x \to \infty\,,$$

where x_0 is a constant. [*Hint*: You do not need to find the exact integral of dx/dh.] Sketch the profile of this flow on the original inclined plane and describe what it represents.

(c) Suppose, instead, there is a large semicircular barrier with position defined by $x^2 + y^2 = R^2$, $x < 0$, where $R \gg 1$. The barrier protrudes normal to the plane, and the flow must go round it. Let (r, ϕ) denote polar coordinates with $\phi = 0$ pointing upslope (so $x = -r\cos\phi$), and let $s = r - R$. Assume that, for $-\pi/2 < \phi < \pi/2$, the flow thickness has the approximate form

$$h(r,\phi) = H(\phi)\left(1 - \frac{s}{\Delta(\phi)}\right) \text{ for } 0 < s < \Delta(\phi)\,, \qquad h \approx 1 \text{ for } s > \Delta(\phi)\,,$$

where $H(\phi) \gg 1$ and $1 \ll \Delta(\phi) \ll R$.

By considering the flux in the s-direction, explain why this form is a reasonable assumption provided $H = \Delta\cos\phi$.

Sketch the streamlines of the flow in $x < 0$. Calculate the leading-order approximation to the total flux in the ϕ-direction in $0 < s < \Delta$, and deduce that $H = (4R\cos\phi)^{1/4}$.

(d) State briefly why the form of solution assumed in part (c) is clearly inconsistent as $\overline{\phi} \to 0$, where $\overline{\phi} = \pi/2 - \phi$.

Assume that a transition to another form of solution occurs when $\partial h^3/(R\,\partial\phi)$ is no longer much smaller than $\partial h^3/\partial r$. Use order-of-magnitude estimates to show that this is when $\overline{\phi} = O(R^{-a})$, where $0 < a < 1$ is to be found, and find the corresponding scalings of H and Δ with R.

(e) Describe qualitatively and physically the expected form of the downslope flow in $x > 0$. Include a brief scaling argument for why there is a change of behaviour at $x = O(R^2)$, but do not attempt any detailed calculations.

END OF PAPER