UNIVERSITY OF CAMBRIDGE

MAMA/324, NST3AS/324, MAAS/324

MAT3

MATHEMATICAL TRIPOS Part III

Friday, 2 June, 2023 1:30 pm to 4:30 pm

PAPER 324

QUANTUM COMPUTATION

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

Attempt no more than **THREE** questions.
There are **FOUR** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(a)

(i) For a quantum unitary circuit $\mathcal{C}$ on n qubits followed by a projective measurement of $k \leqslant n$ qubits in the computational basis, define the notions of strong and weak classical simulation. State the Extended Gottesman Knill Theorem.

(ii) Let $\mathcal{C}$ be an adaptive Clifford circuit acting on $n+t$ qubits initialized in $|\sigma\rangle \otimes |\rho\rangle$ where $|\sigma\rangle$ is a stabilizer state on n qubits and $|\rho\rangle$ is any state on t qubits. Some of the Z measurements in $\mathcal{C}$ are postselected to outcome $+1$. Show that this circuit can be weakly simulated by a corresponding Pauli-Based Computational (PBC) circuit in which some of the Pauli measurements are postselected to outcome $+1$.

(b)

(i) Consider the circuit depicted in Figure 1 below acting on the input state $|A\rangle^{\otimes 2}|0\rangle$; for $\sigma_i \in \{\mathbb{I}, X, Y, Z\}$, $i = 1, 2$ denotes a single-qubit Pauli operation and the computational basis measurement on the third qubit yields the outcome $s \in \{0, 1\}$. Write down the expression for $|\Psi_a\rangle$. Consider the PBC depicted in Figure 2, where $P_1 = \sigma_1 \otimes \sigma_2$. Express $|\Psi_b\rangle$ in terms of the eigenstates of P_1. Compare $|\Psi_a\rangle$ to $|\Psi_b\rangle$.

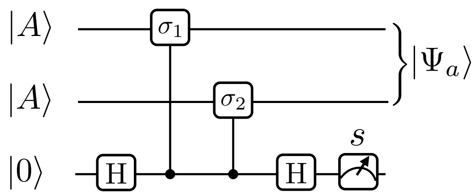

Figure 1

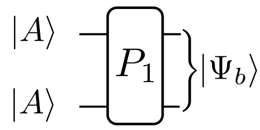

Figure 2

(ii) We wish to implement a two-qubit PBC on input $|A\rangle^{\otimes 2}|0\rangle$ (where $|0\rangle$ is an ancilla qubit) consisting of first measuring $P_1 = Z \otimes \mathbb{I}$ followed by $P_2 = Z \otimes X$ on $|A\rangle^{\otimes 2}$. Let $|\Psi_{out}\rangle$ be the two-qubit output state of the PBC process. In the laboratory we are able to apply single- and two-qubit Clifford operations on the input state and ancilla qubits and perform computational basis measurements on any ancilla qubits. Show how we can generate $|\Psi_{out}\rangle$ in the laboratory with the help of a single ancilla qubit (initialized in state $|0\rangle$) thus implementing this instance of the PBC.

2

(a) Describe the Harrow-Hassidim-Lloyd (HHL) algorithm for estimating $\langle x|M|x\rangle$, where the associated system of equations is given by $A\underline{x} = \underline{b}$, with $\underline{x}, \underline{b} \in \mathbb{C}^N$, and A, M are Hermitian. Include a clear statement of any results that you use. You may assume that $N = 2^n$, the usually prescribed conditions on the ingredients $A, \underline{b}$ and M have been fulfilled, and that any operations used are error-free.

(b) State the usually prescribed conditions on vector $\underline{b}$ in (a) which ensure that the associated normalized state $|b\rangle$ in the HHL algorithm can be efficiently prepared.

(c) Consider a probability distribution function p of a random variable X which takes values on $[0, 1]$. We wish to prepare a state that represents the $N-$point discretization of p, where $N = 2^n$. For $m > 1$ we partition $[0, 1]$ into 2^m intervals labelled $i = 0, \ldots, 2^m - 1$ from left to right with the i-th interval being $[x_L^i, x_R^i]$.

Suppose we have the state $|\psi_m\rangle = \sum_{i=0}^{2^m-1} \sqrt{p_i^{(m)}}|i\rangle$, where $p_i^{(m)}$ is the probability for the random variable X to lie in the interval i. Consider $f(i) = A_i/B_i$, where $A_i = \int_{x_L^i}^{(x_R^i + x_L^i)/2} p(x)dx, B_i = \int_{x_L^i}^{x_R^i} p(x)dx$.

(i) Assuming that each A_i, B_i can be computed efficiently, describe how to generate $|\tilde{\psi}_m\rangle = \sum_{i=0}^{2^m-1} \sqrt{p_i^{(m)}}|i\rangle|\theta_i\rangle$, from $|\psi_m\rangle$ (adding ancillary qubits), where $\theta_i = \arccos(\sqrt{f_i})$.

(ii) Describe how to perform a controlled rotation of angle θ_i which maps $\sqrt{p_i^{(m)}}|i\rangle|\theta_i\rangle \to \sqrt{p_i^{(m)}}|i\rangle|\theta_i\rangle(\cos(\theta_i)|0\rangle + \sin(\theta_i)|1\rangle)$ as an $O(\text{poly}\log(N))-$ sized circuit of 1- and 2-qubit gates. You may ignore any issues of precision that arise.

(iii) Using the above, describe an algorithm for generating $|\psi_n\rangle = \sum_{i=0}^{2^n-1} \sqrt{p_i^{(n)}}|i\rangle$ from $|\psi_1\rangle$, thereby generating the desired N-point discretization state for p.

3

(i) Suppose that G is a linear subspace of state space $\mathcal{H}$ and let $G^{\perp}$ be the orthogonal complement of G in $\mathcal{H}$. Let $|\psi\rangle$ be any state in $\mathcal{H}$. Define the operator $I_{|\psi\rangle}$ of reflection in the hyperplane orthogonal to $|\psi\rangle$, and the operator I_G, of reflection in the subspace $G^{\perp}$. In terms of these, state and prove the Amplitude Amplification (AA) Theorem for a rotation R that you should define.

(ii) Suppose we are given a quantum oracle U_f for a function $f : \{0,1\}^n \to \{0,1\}$. For $\mathcal{H}$ being the state space of n qubits let $G = \text{span}\{|x\rangle : f(x) = 1\}$ and $G^{\perp} = \text{span}\{|x\rangle : f(x) = 0\}$. We say that x is "good" if $f(x) = 1$, and "bad" if $f(x) = 0$. Suppose we have the n-qubit starting state $|\psi_{st}\rangle = \sin(\theta)|g\rangle + \cos(\theta)|b\rangle$, where $|g\rangle \in G, |b\rangle \in G^{\perp}$ and the value of $\theta \in (0, \pi/2)$ is known. Show how the AA theorem may be used to provide a process (possibly including extra ancilla qubits) which will map $|\psi_{st}\rangle$ to a state upon which a final computational basis measurement will yield a good x with certainty.

(iii) Suppose now that the value of $\theta \in (0, \pi/2)$ in (ii) is unknown but we can still prepare the state $|\psi_{st}\rangle$. Let $n^* \in \mathbb{N}$ be the least number of iterations of the R in the AA theorem that is needed to rotate $|\psi_{st}\rangle$ closest to its good projection $|g\rangle$. We will employ the following strategy to find a good x: for each $k = 0, 1, \ldots$ in turn we prepare $|\psi_{st}\rangle$ and apply R k times before making a final measurement. We continue until we obtain a good x. Suppose this occurs for $k = k^*$. Compute the expected number of oracle calls needed to obtain a good x. Show that $Prob[k^* \geqslant n^*] \geqslant const > 0$.

(iv) For the scenario of (iii) suppose now that $n^* = 2^N, N \in \mathbb{N}$ and we modify the strategy in (iii) as follows. Instead of considering each $k = 0, 1, 2, 3\ldots$ in turn we use only the values $k = 2^i$, $i = 0, 1, 2, \ldots$ in turn. Compute the expected number of oracle calls in this case needed to obtain a good x and compare it to the result of (iii).

4

(a)

(i) Consider the n-qubit Pauli group $\mathcal{P}_n = \{kP_1 \otimes \ldots \otimes P_n : k = \pm 1, \pm i;$ and each $P_i \in \{X, Y, Z, \mathbb{I}\}$. Let $S = \{\mathbb{I}, Z_1Z_2, Z_2Z_3, Z_1Z_3\}$. Determine the 3-qubit subspace V_S which is stabilized by the elements of S.

(ii) Let $U = CNOT_{12}$. Determine the elements of $\mathcal{P}_2$ obtained by the following conjugations: $UX_1U^\dagger$, $UX_2U^\dagger$,$UZ_1U^\dagger$,$UZ_2U^\dagger$. Suppose that $\tilde{V}$ is another 2-qubit unitary operator that transforms Z_1, Z_2, X_1, X_2 under conjugation in the same way as U above. Show that $\tilde{V} = U$ up to an overall phase.

(iii) Consider the following two-qubit operators $V_n = \exp(\frac{i\pi}{n} X \otimes X)$, $W_n = (\mathbb{I} \otimes H)V_n(\mathbb{I} \otimes H)$, where H is the Hadamard gate, and $n = 1, 2$. Find A_n such that $V_nW_n = \exp(iA_n)$. Is V_nW_n a Clifford operation?

(b) For any graph $G = (V, E)$, introduce $|V|$ qubits labelled by the vertices of G. Introduce also a corresponding so-called graph state given by $|G\rangle = \prod_{(i,j)\in E} CZ_{ij}|+\rangle^{\otimes|V|}$.

(i) Consider two graphs $G_A = (V_A, E_A)$, where $V_A = \{1, 2, 3\}$ and $E_A = \{(1, 2), (2, 3)\}$, and $G_B = (V_B, E_B)$, where $V_B = \{1, 2, 3\}$ and $E_B = \{(1, 2), (2, 3), (3, 1)\}$. Compute their corresponding graph states $|G_A\rangle$ and $|G_B\rangle$.

(ii) To each vertex v of G we can associate an operator $S_v = X_v \prod_{u\in N(v)} Z_u$, where $N(v)$ is the set of vertices which are adjacent to v. Show that for $G = G_A$ and G_B, the corresponding operators S_v commute for each vertex $v \in \{1, 2, 3\}$.

(iii) Let $|\alpha\rangle$, $|\beta\rangle$ be states where $|\beta\rangle = U|\alpha\rangle$ and $|\alpha\rangle$ is stabilized by S. Show that $|\beta\rangle$ is stabilized by $USU^\dagger$. You may assume without proof that the graph state $|G\rangle$ can also be defined as the simultaneous +1 eigenstate of the $|V|$ stabilizer operators $\{S_v\}_{v\in V}$: $S_v|G\rangle = |G\rangle$. Show how the stabilizer group of $|G_B\rangle$ can be obtained via a suitable mapping of the stabilizer group of $|G_A\rangle$.

END OF PAPER