UNIVERSITY OF CAMBRIDGE

MAMA/321, NST3AS/321, MAAS/321

MAT3

MATHEMATICAL TRIPOS Part III

Wednesday, 7 June, 2023 1:30 pm to 3:30 pm

PAPER 321

DYNAMICS OF ASTROPHYSICAL DISCS

Before you begin please read these instructions carefully

Candidates have TWO HOURS to complete the written examination.

Attempt no more than **TWO** questions.
There are **THREE** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

In the model of the compressible shearing sheet, a self-gravitating disc has surface density $\Sigma(x, y, t)$ and velocity $\mathbf{u}(x, y, t)$ satisfying the equation of mass conservation,

$$\frac{\partial \Sigma}{\partial t} + \boldsymbol{\nabla} \cdot (\Sigma \mathbf{u}) = 0\,,$$

and the equation of motion,

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \boldsymbol{\nabla} \mathbf{u} + 2\boldsymbol{\Omega} \times \mathbf{u} = -\boldsymbol{\nabla} \Phi_{\mathrm{t,m}} - \boldsymbol{\nabla} \Phi_{\mathrm{d,m}} - \frac{1}{\Sigma} \boldsymbol{\nabla} P + \frac{1}{\Sigma} \boldsymbol{\nabla} \cdot \mathbf{T}\,,$$

where $\Phi_{\mathrm{t,m}}$ is the tidal potential in the midplane $z = 0$, $\Phi_{\mathrm{d,m}}$ is the gravitational potential of the disc in the midplane, and the viscous stress tensor $\mathbf{T}$ is given in terms of the kinematic shear viscosity ν and bulk viscosity ν_{b} by

$$T_{ij} = \nu \Sigma \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) + \left(\nu_{\mathrm{b}} - \tfrac{2}{3} \nu \right) \Sigma \frac{\partial u_k}{\partial x_k} \, \delta_{ij}\,.$$

Assume that the two-dimensional pressure is given by a barotropic relation $P = P(\Sigma)$ and that the viscosity coefficients are also functions of the surface density.

(a) Explain how $\Phi_{\mathrm{d,m}}$ is related to Σ within this local model of a disc, when the thickness of the disc is neglected. Give an explicit expression for the horizontal Fourier transform of $\Phi_{\mathrm{d,m}}$ in terms of the Fourier transform of Σ.

(b) Write down an expression for $\Phi_{\mathrm{t,m}}$ in terms of the angular velocity Ω, assuming that the disc is Keplerian. Define an appropriate steady solution of this local model representing a uniform disc in circular orbital motion. Why does the viscous stress in this state not lead to an accretion flow?

(c) Formulate the linearized equations governing small perturbations $\propto \exp(ik_x x + \lambda t)$, and independent of y, to this steady solution. Hence derive the dispersion relation

$$(\lambda + \nu k^2) \Big\{ \lambda \left[\lambda + \left(\nu_{\mathrm{b}} + \tfrac{4}{3} \nu \right) k^2 \right] - 2\pi G \Sigma k + v_{\mathrm{s}}^2 k^2 \Big\} + \lambda \Omega^2 + 3\beta \Omega^2 \nu k^2 = 0\,,$$

giving the growth rate λ of axisymmetric waves in terms of their wavenumber $k = |k_x|$, where v_{s} is the adiabatic sound speed and

$$\beta = \frac{d \ln(\nu \Sigma)}{d \ln \Sigma}\,.$$

(d) First simplify the dispersion relation in the case of an inviscid disc. Show that such a disc is unstable to axisymmetric modes when

$$Q = \frac{v_{\mathrm{s}} \Omega}{\pi G \Sigma} < 1\,.$$

[QUESTION CONTINUES ON THE NEXT PAGE]

(e) Now consider the case of a viscous disc, in which different modes of instability are possible. You may assume that a necessary condition for the roots of the real cubic equation

$$\lambda^3 + a\lambda^2 + b\lambda + c = 0$$

all to have $\mathrm{Re}(\lambda) \leqslant 0$ is that $c \geqslant 0$. Deduce that a viscous disc is unstable in the limit $k \to 0$ in the case $\beta < 0$. How is this result related to an analysis of the diffusion equation for the viscous evolution of a Keplerian disc? If instead $\beta > 0$, show that the disc is unstable if

$$3\beta\Omega^2 - 2\pi G\Sigma k + v_{\mathrm{s}}^2 k^2 < 0$$

for some $k > 0$, and show further that this occurs when

$$Q < \frac{1}{\sqrt{3\beta}}\,.$$

2

Consider the gravitational interaction of a disc with a massive satellite in the local approximation. The equations of 2D gas dynamics for an ideal, non-self-gravitating disc are

$$\frac{\partial \Sigma}{\partial t} + \boldsymbol{\nabla} \cdot (\Sigma \mathbf{u}) = 0$$

and

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \boldsymbol{\nabla} \mathbf{u} + 2\boldsymbol{\Omega} \times \mathbf{u} = -\boldsymbol{\nabla} \Phi_{\mathrm{t,m}} - \boldsymbol{\nabla} \Psi - \frac{1}{\Sigma} \boldsymbol{\nabla} P \,,$$

where $\Phi_{\mathrm{t,m}} = -\Omega S x^2$ is the tidal potential in the midplane and $\Psi(x, y, t)$ is the gravitational potential due to the satellite. Assume that the two-dimensional pressure is given by a barotropic relation $P = P(\Sigma)$. In the absence of the satellite potential, the basic state is a uniform disc with the orbital shear flow $\mathbf{u} = -Sx \, \mathbf{e}_y$.

(a) Derive the linearized equations

$$\begin{aligned}
\frac{D\Sigma'}{Dt} &= -\Sigma \left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} \right) \,, \\
\frac{Dv_x}{Dt} - 2\Omega v_y &= -\frac{\partial \Psi}{\partial x} - \frac{v_{\mathrm{s}}^2}{\Sigma} \frac{\partial \Sigma'}{\partial x} \,, \\
\frac{Dv_y}{Dt} + (2\Omega - S) v_x &= -\frac{\partial \Psi}{\partial y} - \frac{v_{\mathrm{s}}^2}{\Sigma} \frac{\partial \Sigma'}{\partial y}
\end{aligned}$$

for the disturbance generated by the satellite, where Σ' and $\mathbf{v}$ are the perturbations of surface density and velocity, v_{s} is the adiabatic sound speed and

$$\frac{D}{Dt} = \frac{\partial}{\partial t} - Sx \frac{\partial}{\partial y} \,.$$

(b) Show that the quantity

$$f' = \frac{1}{\Sigma} \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} \right) - (2\Omega - S) \frac{\Sigma'}{\Sigma^2}$$

satisfies $Df'/Dt = 0$, and give a physical interpretation of this result.

[QUESTION CONTINUES ON THE NEXT PAGE]

(c) When the initial conditions are such that f' vanishes, you may assume that the equations of part (a) can be combined into the form

$$\left(\frac{D^2}{Dt^2} + \Omega_r^2 - v_{\rm s}^2\nabla^2\right) v_y = (2\Omega - S)\frac{\partial\Psi}{\partial x} - \frac{D}{Dt}\frac{\partial\Psi}{\partial y}\,. \qquad (\dagger)$$

where Ω_r is the radial (epicyclic) frequency. [You are *not* required to derive (†).]

When the disc is unforced ($\Psi = 0$), show that there are solutions of (†) in the form of shearing density waves,

$$v_y = \mathrm{Re}\left[\tilde{v}_y(t)\, e^{i\mathbf{k}(t)\cdot\mathbf{x}}\right],$$

provided that the wavevector $\mathbf{k}(t)$ evolves in time in a way that you should determine, and provided that $\tilde{v}_y(t)$ satisfies the ordinary differential equation (ODE)

$$\frac{d^2\tilde{v}_y}{dt^2} + g(t)\tilde{v}_y = 0\,, \qquad (\ddagger)$$

where $g(t)$ is a function that you should determine explicitly.

(d) When the disc is forced by a satellite on a circular orbit, explain why the potential Ψ can be assumed to be independent of time.

Consider a single Fourier component $\Psi = \psi(x)\exp(ik_y y)$ of the satellite potential, and assume that the disc's response to this is also of the form $v_y = v(x)\exp(ik_y y)$. From (†), derive an ODE for $v(x)$ describing the radial structure of the density waves forced by the satellite. [You are *not* required to work out the form of $\psi(x)$.]

(e) Without solving the equation, show that the homogeneous (unforced) version of the ODE in part (d) has an oscillatory character for sufficiently large $|x|$, and show that the orbital motion of the disc relative to the satellite is supersonic in the oscillatory regions.

(f) Take the Fourier transform of the ODE in part (d) with respect to x and derive an ODE for $\tilde{v}(k_x)$, where

$$\tilde{v}(k_x) = \int_{-\infty}^{\infty} v(x)\, e^{-ik_x x}\, dx.$$

Comment on the relation of this ODE to (‡), bearing in mind the time-dependence of the shearing wavevector in part (c).

[*Hint:* For suitably well-behaved $u(x)$,

$$\int_{-\infty}^{\infty} xu(x)\, e^{-ik_x x}\, dx = i\frac{d}{dk_x}\int_{-\infty}^{\infty} u(x)\, e^{-ik_x x}\, dx\,.]$$

3

(a) Derive an expression for the angular frequency Ω_z of small vertical oscillations about a circular orbit in the midplane of an axisymmetric potential $\Phi(r, z)$ with reflectional symmetry. Show that Ω_z is equal to the orbital frequency in the case of a spherically symmetric potential, and explain the reason for this.

(b) A non-self-gravitating perfect gas in the local model of an astrophysical disc satisfies the three-dimensional equations

$$\frac{\partial \rho}{\partial t} + \mathbf{u} \cdot \boldsymbol{\nabla} \rho + \rho \boldsymbol{\nabla} \cdot \mathbf{u} = 0\,,$$

$$\frac{\partial p}{\partial t} + \mathbf{u} \cdot \boldsymbol{\nabla} p + \gamma p \boldsymbol{\nabla} \cdot \mathbf{u} = 0\,,$$

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \boldsymbol{\nabla} \mathbf{u} + 2\boldsymbol{\Omega} \times \mathbf{u} = -\boldsymbol{\nabla} \Phi_{\mathrm{t}} - \frac{1}{\rho} \boldsymbol{\nabla} p\,,$$

where γ is the adiabatic exponent and $\Phi_{\mathrm{t}} = -\Omega S x^2 + \frac{1}{2}\Omega_z^2 z^2$ is the tidal potential. For a steady solution $\mathbf{u} = -Sx\,\mathbf{e}_y$, $\rho = \rho(z)$, $p = p(z)$, derive the equation of vertical hydrostatic equilibrium and show that the surface density Σ and the vertically integrated pressure P are related by $P = \Sigma H^2 \Omega_z^2$, if the scaleheight H is defined such that H^2 is the mass-weighted average of z^2.

(c) Using P, Σ and H as natural units, explain how to express the hydrostatic vertical structure in a dimensionless form using dimensionless variables (denoted by $\tilde{}$) that satisfy the equation

$$\frac{d\tilde{p}}{d\tilde{z}} = -\tilde{\rho}\tilde{z}$$

and the normalization conditions

$$\int \tilde{\rho}\, d\tilde{z} = \int \tilde{p}\, d\tilde{z} = 1\,,$$

where the integrals are over the full vertical extent of the disc. Give one explicit example of a solution of this dimensionless problem.

(d) Show that the vertical motion of a dust grain, subject to gas drag with a constant stopping time τ, in a gas disc that is in vertical hydrostatic equilibrium is similar to a damped harmonic oscillator. Describe the motion of the grain qualitatively.

[QUESTION CONTINUES ON THE NEXT PAGE]

(e) Now consider a dynamical situation in which the gas disc is undergoing a uniform expansion or contraction in the vertical direction, such that the scaleheight is $H(t)$ and (apart from the orbital shear flow) the motion is independent of x and y. Let $\zeta = z/H(t)$ be a Lagrangian variable that is constant following the motion of the gas (i.e. $D\zeta/Dt = 0$). Verify that the equations of gas dynamics are satisfied when

$$\rho = \frac{\Sigma}{H(t)}\,\tilde{\rho}(\zeta)\,, \qquad p = \frac{P(t)}{H(t)}\,\tilde{p}(\zeta)\,, \qquad \mathbf{u} = -Sx\,\mathbf{e}_y + \frac{dH}{dt}\,\zeta\,\mathbf{e}_z$$

(where $\tilde{\rho}$ and $\tilde{p}$ are the same dimensionless functions as in the hydrostatic solution in part (c), but with $\tilde{z}$ replaced by ζ), provided that $H(t)$ and $P(t)$ satisfy

$$\frac{d^2H}{dt^2} + \Omega_z^2 H = \frac{P}{\Sigma H} = CH^{-\gamma}\,,$$

where C and Σ are constants.

(f) Hence obtain an expression, in terms of γ and Ω_z, for the angular frequency of small oscillations of the scaleheight H about its equilibrium value.

END OF PAPER