UNIVERSITY OF CAMBRIDGE

MAMA/320, NST3AS/320, MAAS/320

MAT3

MATHEMATICAL TRIPOS Part III

Monday, 12 June, 2023 1:30 pm to 3:30 pm

PAPER 320

MODERN STELLAR DYNAMICS

Before you begin please read these instructions carefully

Candidates have TWO HOURS to complete the written examination.

Attempt no more than **TWO** questions.
There are **THREE** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

During the formation of a galaxy, the gas is slowly accreted into the potential well of a dark matter halo and is converted into stars. The most massive stars have short lifetimes and soon explode as supernovae, ejecting most of their mass with high velocity, so that the gravitational potential suddenly becomes shallower, and the orbits of remaining stars and dark matter particles expand or may even become unbound. This process, especially when repeated many times in consecutive starbursts, is believed to be responsible for a gradual erosion of cuspy density profiles of dark haloes (Pontzen & Governato 2012). In this question, we consider this process in a very much simplified setup.

Consider a spherical isotropic stellar system with some density profile $\rho(r)$ initially embedded in a Kepler potential $\Phi_0(r) = -GM_0/r$. Then the mass generating this potential is suddenly reduced to $M_1 = 0.5\, M_0$.

(a) Write down expressions relating the orbital energy E, angular momentum L, semimajor axis $a \equiv (r_- + r_+)/2$, and eccentricity $e \equiv (r_+ - r_-)/(r_+ + r_-)$, where r_-, r_+ are the peri- and apocentre radii.

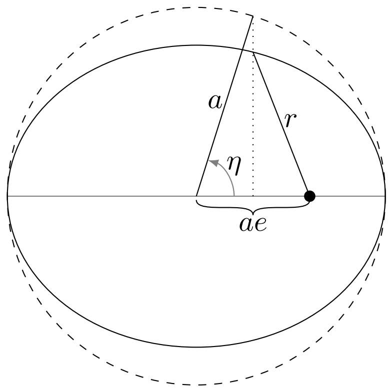
(b) Derive the eccentricity distribution of stars $n(e)$, defined such that $n(e)\,\mathrm{d}e$ is the fraction of stars with eccentricity in the interval $(e, e + \mathrm{d}e)$. Hint: in the isotropic case, the distribution of stars in angular momentum at a fixed energy is $n(L) \propto L$.

(c) For an orbit with a given semimajor axis a and eccentricity e, determine the probability that a star will become unbound after the sudden decrease of the potential (equivalently, the fraction of stars in a phase-mixed population that will become unbound). In which part of the orbit are the stars more likely to become unbound: close to the pericentre or to the apocentre?

(d) Determine the overall unbound fraction, after averaging the above expression over the eccentricity distribution of all stars.

(e) How will the results change if the mass decreases from M_0 to M_1 very slowly? Compute the unbound fraction in this case, and determine whether the distribution of remaining stars will become more radially or tangentially anisotropic, or stay isotropic.

Useful expressions:



An elliptical orbit can be parametrised by the angle η (called "eccentric anomaly"): it is related to the radial phase angle θ_r ("mean anomaly") by
$\theta_r = \eta - e\,\sin\eta$,
and the distance from the origin is given by
$r = a\,(1 - e\,\cos\eta)$,
where a is the semimajor axis and e is the eccentricity.

Another possibly useful expression is

$$\int_a^b \mathrm{d}x\, \frac{\sqrt{(x-a)(b-x)}}{x} = \pi\left[\frac{a+b}{2} - \sqrt{ab}\right].$$

2

In this course, we often considered examples of scale-free density, potential or distribution function (DF) profiles, but remarked that these are not physically realistic (have divergent properties at either small or large values of argument). In this question, we look at a slightly more complicated model that consists of two scale-free components (An & Evans 2006).

Consider a spherical galaxy model with the following potential profile:

$$\Phi(r) = -\frac{GM}{b + \left(\sqrt{b} + \sqrt{r}\right)^2}.$$

(a) Show that the corresponding density profile can be written in the form

$$\rho(r) = \sum_{k=1}^{2} A_k \, r^{-\gamma_k} \left[-\Phi(r) \right]^{p_k},$$

determine the power-law indices γ_k, p_k and the coefficients A_k. Derive the asymptotic behaviour of the density at small and large radii and demonstrate that the model is physically realistic (its total mass is finite and the circular velocity tends to zero at origin).

(b) Consider a DF for a spherical anisotropic system in the form

$$f(E, L) = f_0 \, L^{-2\beta} \, |E|^n,$$

where $E < 0$ is the orbital energy and L is the magnitude of the total angular momentum. Show that the density profile generated by this DF can be written in the same form

$$\rho(r) = A \, r^{-\gamma} \left[-\Phi(r) \right]^p,$$

in any potential, determine the corresponding indices and coefficients. By comparing this expression with the one from (a), demonstrate that the DF of that model can be written as a sum of two such components.

(c) Determine the asymptotic behaviour of the velocity anisotropy coefficient $\hat{\beta} \equiv 1 - \sigma_\mathrm{t}^2/(2\sigma_\mathrm{r}^2)$ at small and large radii for this model, where σ_r and σ_t are the radial and tangential velocity dispersions (the latter is the sum of dispersions in both components of tangential velocity).

Useful expression:

$$\int_0^1 x^a \, (1-x)^b \, \mathrm{d}x = \mathrm{B}(a+1, b+1) \equiv \frac{\Gamma(a+1)\,\Gamma(b+1)}{\Gamma(a+b+2)}.$$

3

Consider a satellite galaxy on a circular orbit around the host galaxy, where the density profiles of both systems are given by $\rho_{\mathrm{h,s}}(r) = A_{\mathrm{h,s}}\, r^{-1}$, and the normalization coefficient A_{s} for the satellite is much lower than for the host A_{h}. The satellite faces two predicaments:

(1) it experiences dynamical friction according to the Chandrasekhar's formula

$$\frac{\mathrm{d}v}{\mathrm{d}t} = -\frac{4\pi\, G^2\, M_{\mathrm{s}}\, \rho_{\mathrm{h}}\, \ln\Lambda}{v^2}\, \mathcal{K},$$

where $\ln\Lambda$ is the Coulomb logarithm and $\mathcal{K}$ is a numerical coefficient depending on the ratio between the satellite's velocity and the velocity dispersion of the host galaxy; both of these coefficients are assumed to be constant.

(2) it suffers from tidal forces, stripping away the material beyond the tidal radius r_{t}. For simplicity, we may assume that the density profile of the satellite is sharply truncated at the tidal radius and is unchanged inside it, hence its total mass M_{s} is finite.

As the satellite sinks towards the centre of the host galaxy due to dynamical friction, tidal forces become stronger, the tidal radius shrinks and the satellite mass decreases, thus reducing the efficiency of friction. In this question, we examine whether the satellite is destined to be completely disrupted in a finite time as it falls into the centre of the host galaxy.

(a) Determine the tidal radius r_{t} of the satellite as a function of its orbital radius r_{o} (assuming a circular orbit) and justify the applicability of the resulting expression.

(b) Write down and solve the differential equation for the time evolution of satellite's orbital radius r_{o} and its mass. Hint: the acceleration given by the Chandrasekhar formula can be converted into the time derivative of the orbital angular momentum, and the latter expressed in terms of orbital radius (for the given host galaxy potential). We assume that the satellite stays on a circular orbit.
Does the satellite reach zero radius and mass in a finite time?

(c) How would the decay rate change if the satellite mass stayed constant (i.e. ignoring the tidal stripping)?

(d) How would the results change if the host galaxy had a singular isothermal density profile $\rho \propto r^{-2}$, but the satellite density still followed the r^{-1} profile truncated at the tidal radius? Qualitatively explain the difference from case (b).

END OF PAPER