UNIVERSITY OF CAMBRIDGE

MAMA/319, NST3AS/319, MAAS/319

MAT3

MATHEMATICAL TRIPOS Part III

Friday, 9 June, 2023 1:30 pm to 3:30 pm

PAPER 319

UNBOUNDED OPERATORS AND SEMIGROUPS

Before you begin please read these instructions carefully

Candidates have TWO HOURS to complete the written examination.

Attempt **ALL** parts of the question.
There is **ONE** question in total.

In doing a given part of the question you may use assertions in preceding parts even if you did not complete that part.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

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(a) Let X be a Banach space with norm $\|\cdot\|$. State the Hille-Yosida theorem for C_0 semigroups $\{U\}_{t\geqslant 0}$ which obey the bound $\|U(t)u\| \leqslant Me^{\omega t}\|u\| \, \forall u \in X$. Include in your answer a definition of the generator $A \in \mathcal{G}(M,\omega)$, and of its domain, and give the precise characterization of the set of generators of such semigroups.

(b) Prove that the domain $\mathrm{Dom}\,(A)$ is a dense subspace of X.

(c) Consider the initial value problem (for real-valued $u = u(t,x)$):

$$u_{tt} - u_{xx} + u = 0 \qquad u(0,x) = u_0(x)\,,\; u_t(0,x) = u_1(x)\,, \tag{1}$$

with periodic boundary conditions $u(t,x+2\pi) = u(t,x)$. Formulate (1) as an abstract evolution equation $\dot{Z} = AZ$ in the form

$$\partial_t \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -B^2 & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = A \begin{pmatrix} u \\ v \end{pmatrix} \tag{2}$$

on the Hilbert space $\mathcal{H}$ of pairs $Z = (u,v)$ of 2π-periodic L^2 functions with

$$\|Z\|_{\mathcal{H}}^2 = \|(u,v)\|_{\mathcal{H}}^2 = \int_{-\pi}^{+\pi} u_x^2 + u^2 + v^2 \, dx \; < +\infty\,.$$

Using Fourier series, or otherwise, describe $\mathcal{H}$ precisely and specify the domain of A in your formulation.

(d) State the Lumer-Phillips theorem, and hence verify that your answer to (c) gives rise to a one-parameter unitary group U on $\mathcal{H}$.

(e) Let $f = f(t,x) = f(t,x+2\pi)$ be a given real-valued function. Write the inhomogeneous initial value problem

$$u_{tt} - u_{xx} + u = f \qquad u(0,x) = u_0(x)\,,\; u_t(0,x) = u_1(x)\,, \tag{3}$$

in first order form $\dot{Z} = AZ + F\,,\; Z(0) = Z_0\,,$ where you should specify how Z_0, F are determined by u_0, u_1, f. Hence formulate a notion of mild solution for the initial value problem as an integral equation

$$Z(t) = U(t)Z_0 + \int_0^t U(t-s)F(s)\,ds\,. \tag{4}$$

Show that if $s \mapsto F(s) \in \mathcal{H}$ and $s \mapsto AF(s) \in \mathcal{H}$ are continuous for $s \geqslant 0$, then $Z \in C^1([0,\infty);\mathcal{H})$ and

$$\|Z(t)\|_{\mathcal{H}}^2 \leqslant \|Z(0)\|_{\mathcal{H}}^2 + 2\int_0^t |(Z(s),F(s))_{\mathcal{H}}| ds\,.$$

(f) For the nonlinear initial value problem

$$u_{tt} - u_{xx} + u = u^2 \qquad u(0,x) = u_0(x)\,,\; u_t(0,x) = u_1(x)\,, \tag{5}$$

prove, using the contraction mapping theorem or otherwise, that given $Z(0) = (u_0,u_1) \in \mathcal{H}$ there is a mild solution $t \mapsto Z(t)$ to (5) for $t \in [0,T]$, stating exactly what you mean by this. Explain whether or not the solution (i) will be differentiable, and (ii) will exist for all time i.e. for $t \in [0,+\infty)$.

END OF PAPER