UNIVERSITY OF CAMBRIDGE

MAMA/317, NST3AS/317, MAAS/317

MAT3

MATHEMATICAL TRIPOS Part III

Thursday, 1 June, 2023 1:30 pm to 4:30 pm

PAPER 317

STRUCTURE AND EVOLUTION OF STARS

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

Attempt no more than **THREE** questions.
There are **FOUR** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Consider a spherically symmetric star of total mass M and radius R. Assume that energy transport is by radiative diffusion in a perfect gas. The mean molecular weight μ is uniform. Radiation pressure can be neglected.

Write down the equations of stellar structure for the above specified star.

Assume that the density distribution $\varrho(r)$ in this star is given by

$$\varrho = \varrho_{\rm c} \left(1 - \frac{r}{R}\right) ,$$

where $\varrho_{\rm c}$ is the central density of the star and r is the distance from the centre of the star.

(i) Find m_r the mass within radius r, the pressure $P(r)$ and the temperature $T(r)$ at radius r. Find the central density $\varrho_{\rm c}$, the central pressure $P_{\rm c}$ and the central temperature $T_{\rm c}$ in terms of R and M.

(ii) Assume now that all the energy is generated at the centre of the star, so that the luminosity at radius r, $L_r = L = \text{const}$ outside a very small region near $r = 0$.

Consider the star's structure and use the appropriate stellar structure equation at $r = 0.5\,R$ to derive an expression for the luminosity L in terms of M and R for two types of opacity that might be dominant in the star's material.

First, assume that Kramer's opacity is dominant in the stellar material and derive

$$L = L(M, R) = C_{\rm K} M^{\alpha} R^{\beta} .$$

Determine α and β and find an expression for $C_{\rm K}$.

Secondly, find $L = L(M, R) = C_{\rm es} M^{\delta} R^{\gamma}$ for the case when electron-scattering opacity is dominant. Determine δ, γ and $C_{\rm es}$.

Your results should show that for electron-scattering opacity, the luminosity L is independent of the radius R and increases with mass M less steeply than for Kramer's opacity.

2

Consider a spherically symmetric star in radiative equilibrium. The equation of state is given by a mixture of a perfect gas and radiation, so the pressure $P = P_{\rm g} + P_{\rm rad}$. The mean molecular weight μ for this star decreases inwards according to the law

$$\mu = \mu_1 \, T(r)^{-s}$$

where $T(r)$ is the temperature of stellar material, r is the distance from the centre of the star, s is a non negative constant and μ_1 is a positive constant.

Let ϱ be the density of stellar material, m_r be the mass within radius r, M the total mass, L_r the star's luminosity at radius r, L the surface luminosity and $\beta = P_{\rm g}/P$.

(i) Show that such a star can be a polytrope with $P = K\,\varrho^{1+\frac{1}{n}}$. Find $K = K(\beta)$ and the polytropic index n for this star.

What is a sufficient condition on β for the star to be a polytrope?

Show that the opacity κ for this star is

$$\kappa\,\eta = 4\pi c G(1-\beta)\frac{M}{L}\,, \qquad \text{where} \qquad \eta = \frac{L_r}{m_r}\,\frac{M}{L}.$$

(ii) Derive the Lane-Emden equation for polytropes of index n using the standard polytropic dimensionless variables θ and ξ, where

$$\varrho = \lambda\,\theta^n \qquad \text{and} \qquad r = \alpha\,\xi,$$

specifying λ and α. Describe the boundary conditions.

Find the mass interior to ξ and the total mass of this configuration.

(iii) Assume now that $s = 0$ so the star described above is chemically uniform. Assume that κ is given by Kramer's opacity and $\eta = const.$ Using properties of polytropes, show that for the considered star

$$L \propto \beta^{7.5}\,M^{11/2}\,R^{-1/2}\,.$$

3

Consider a fully convective star composed of a perfect gas with uniform mean molecular weight μ. Assume that the surface opacity is given by $\kappa = \kappa_0 \, T^{19/2}$, where the surface is defined at an optical depth $\tau = \frac{2}{3}$.

(i) Show that the Hayashi track for this star is such that

$$L \propto M^6 \, T_{\rm eff}^{-44},$$

where L is the star's luminosity, M is the total mass and $T_{\rm eff}$ is effective temperature.

(ii) Suppose that this star is powered by matter settling on to it at a constant rate $\dot{M}$ creating an accretion luminosity $L_{\rm acc} = G \, M \, \dot{M}/R$. Show that the track the star follows on the H–R diagram obeys

$$L \propto \dot{M}^{\frac{3}{4}} \, T_{\rm eff}^{7} \, .$$

(iii) Suppose now that deuterium burning also occurs and that it may be approximated by

$$\epsilon = \epsilon_0 \, \varrho \, T^{15},$$

where ϵ is the energy generation rate per unit mass, ϵ_0 is a constant, ϱ is the density of stellar material and T its temperature. Assume that nuclear energy generation from the deuterium burning $L_{\rm nuc}$ is much smaller than the accretion energy generation $L_{\rm acc}$ and show that

$$\frac{L_{\rm nuc}}{L_{\rm acc}} \propto M^{13} \, \dot{M}^{-7}.$$

Considering the above expressions for L and $\dfrac{L_{\rm nuc}}{L_{\rm acc}}$ deduce that deuterium burning luminosity exceeds the accretion luminosity for some

$$L > L_0 \propto T_{\rm eff}^{\rm d} \, , \qquad \text{where} \qquad d = \frac{964}{43} \, .$$

4

(i) Derive the Ledoux and Schwarzschild criteria for convective stability.

(ii) Consider a mixture of a perfect gas with pressure $P_{\rm g}$ and radiation with pressure $P_{\rm rad}$. The total pressure of this mixture is $P = P_{\rm g} + P_{\rm rad}$ at temperature T.

Let $\nabla_{\rm ad} = \left(\dfrac{\partial \log T}{\partial \log P}\right)_{\rm ad}$ be the adiabatic gradient and $\nabla_{\rm rad} = \left(\dfrac{\partial \log T}{\partial \log P}\right)_{\rm rad}$ be the gradient within the stellar material for radiative transport of energy.

Find $\nabla_{\rm ad}(\beta)$, where $\beta = \dfrac{P_{\rm g}}{P}$, for this mixture. Determine $\nabla_{\rm ad}$ as $\beta \to 1$.

(iii) A star is composed of a perfect gas and radiation pressure can be neglected. The star has a convective core. The opacity of the stellar matter is $\kappa = \kappa_0\, \varrho^a\, T^{-b}$ and the energy generation per unit mass is $\epsilon = \epsilon_0\, \varrho\, T^{\eta}$, κ_0, ϵ_0, a, b and η are constants. Show that if close to the centre $\varrho \propto \varrho_{\rm c}\,(1 - \lambda\, r^2)$, where λ is a positive constant, then

$$L \propto r^3 \left[1 - \frac{3}{5}\left(2 + \frac{2}{3}\eta\right) \lambda\, r^2\right].$$

Find an expression for $\nabla_{\rm rad}$ close to the centre using expansions of L_r, m_r and T obtained with the above form of ϱ close to the centre, in the form $\nabla_{\rm rad} = \nabla_{\rm rad,c}\,(1 - A\, r^2)$ giving A in terms of λ, a, b and η where $\nabla_{\rm rad,c}$ need not be determined.

What condition on $\nabla_{\rm rad\, c}$ is necessary for a convective core?

END OF PAPER