UNIVERSITY OF CAMBRIDGE

MAMA/315, NST3AS/315, MAAS/315

MAT3

MATHEMATICAL TRIPOS Part III

Wednesday, 7 June, 2023 9:00 am to 12:00 pm

PAPER 315

EXTRASOLAR PLANETS: ATMOSPHERES AND INTERIORS

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

Attempt no more than **THREE** questions.
There are **FOUR** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

a. A transiting ultra hot jupiter was found to have an inhomogeneous atmosphere at the day-night terminator. Consider the day-night terminator region to be comprised of two halves: a morning terminator with an isothermal temperature ($T_{\rm m}$) and an evening terminator with an isothermal temperature ($T_{\rm e}$). Derive an expression for the transmission spectrum of the planet. Derive an expression for the average scale height derived from the transmission spectrum as function of $T_{\rm m}$ and $T_{\rm e}$. Make and state any assumptions needed.

b. Consider the dayside atmosphere of a hot jupiter as viewed by a distant observer to be composed of two concentric regions demarcated by a transition angle $\theta_{\rm T}$ from the sub-stellar point, assuming spherical symmetry. The atmosphere is isothermal in each region, with a temperature T_1 for $0 < \theta \leqslant \theta_{\rm T}$ and a temperature T_2 for $\theta_{\rm T} < \theta < \pi/2$.

(i) Starting with the radiative transfer equation, derive an expression for the thermal emission spectrum of the planet. Make and state any assumptions needed.

(ii) The planet was observed during secondary eclipse with the JWST at 20 μm. The eclipse depth was interpreted assuming a homogeneous and isothermal atmosphere. Estimate the derived temperature T as a function of any of the above quantities. Estimate T for $\theta_{\rm T} = \pi/4$, $T_1 = 1500$ K and $T_2 = 1000$ K.

2

a. An exoplanetary atmosphere was observed with the JWST to derive its temperature structure. The observations were sensitive to a pressure range of 10^{-6} - 10 bar. The temperature structure was found to be isothermal for pressures below $P_{\rm t}$ and above $P_{\rm b}$, with $T = T_{\rm t}$ for $P \leqslant P_{\rm t}$ and $T = T_{\rm b}$ for $P \geqslant P_{\rm b}$. At intermediate pressures, $P_{\rm t} < P < P_{\rm b}$, the temperature decreased linearly with altitude.

(i) Derive the temperature as a function of pressure for $P_{\rm t} < P < P_{\rm b}$.

(ii) Derive the temperature gradient with respect to distance in the atmosphere.

b. Consider two adjacent layers of an atmosphere with uniform temperatures T_1 and T_2 and mid-layer altitudes z_1 and z_2 respectively, where $z_1 < z_2$. The corresponding optical depths in the layers are τ_1 and τ_2, respectively, both satisfying $\tau \ll 1$. Consider a beam of radiation incident from below layer 1 with a specific intensity $\mathrm{I}_{\nu,0}$ at normal incidence.

(i) Derive an expression for the emergent specific intensity at the top of layer 2, assuming the layers to be in local thermodynamic equilibrium (LTE). Make and state any assumptions needed.

(ii) Assuming $\mathrm{I}_{\nu,0}$ to be the specific intensity of a blackbody with a temperature T_0, discuss the nature of the emergent spectrum for the following three cases: (1) $T_2 < T_1 < T_0$, (2) $T_2 > T_1 < T_0$, and (3) $T_2 > T_1 > T_0$. Sketch a temperature profile corresponding to each case.

c. Consider the temperature structure of a highly irradiated atmosphere of the following form

$$T^4 = A + B\tau + Ce^{-\beta\tau},$$

where T is the temperature and τ is the optical depth. A, B, C and β are constants and $0 < \beta < 1$.

(i) Derive the condition for a thermal inversion in the observable atmosphere. Discuss how this can be achieved in a typical hot jupiter.

(ii) Derive an expression for the pressure at the radiative convective boundary P_{rc} for such an atmosphere. Estimate P_{rc} for a hot jupiter compared to that for jupiter. Make and state any assumptions required.

3

a. The interior of a spherically symmetric planet of mass $M_{\rm p}$ and radius $R_{\rm p}$ is composed of a core and a mantle of different compositions. The core is of radius $R_{\rm c}$ and uniform density $\rho_{\rm c}$, and is overlaid by the mantle for the rest of the planet with a density profile given by $\rho = \rho_{\rm c} - \rho_{\rm m}(r - R_{\rm c})/R_{\rm c}$, where $\rho_{\rm m}$ is a constant and r is the radial distance from the center of the planet.

(i) Derive the pressure profile in the planetary interior.

(ii) Estimate a limit on the value of $\rho_{\rm m}$ in terms of the other quantities given above.

b. Now consider an isothermal atmosphere on top of the mantle in the above planet, with the base of the atmosphere at a pressure P_0 and temperature T_0. Derive an expression for a characteristic mass of the atmosphere. Make and state any assumptions needed.

c. The total mass and radius of the above planet lie between the theoretical mass-radius curves for a pure silicate planet and a pure water world. The planet is known to reside in the habitable zone around its sun-like star. Discuss the possible interior and atmospheric compositions of the planet. Estimate the ratio of the atmospheric thickness for two such atmospheric compositions. Make and state any assumptions needed.

d. A super-Earth orbits its host star at an orbital separation a and is tidally locked. The host star has a temperature $T_{\rm s}$ and radius $R_{\rm s}$ and emits like a blackbody. Derive an expression for the equilibrium temperature of the planet on its night side. Make and state any assumptions needed. How does the result affect planetary habitability?

4

a. Show how the specific intensity I_ν emitted by a source varies with distance in free space. Show how it is affected by the presence of a cold medium in its path that has coherent isotropic scattering.

b. Show that for a bimolecular reaction in thermochemical equilibrium if the forward reaction rate is known the reverse reaction rate can be determined based on the relevant Gibbs free energies.

c. Consider a simple single-layer atmosphere of a rocky exoplanet which orbits a sun-like star at an orbital separation a. The atmosphere absorbs infrared radiation with an efficiency α and scatters visible radiation with an efficiency β. Derive an expression for the surface temperature.

d. Sketch a mass-radius diagram for spherical bodies between rocky planets and low-mass stars in isolation. Identify and discuss the key transitions. How do hot jupiters deviate from this trend and what could be some potential reasons?

e. Discuss an aspect of giant planetary evolution that made the first direct detections of exoplanets possible. Discuss what governs the luminosity of a giant planet at 10^{10} years of age.

END OF PAPER