UNIVERSITY OF CAMBRIDGE

MAMA/314, NST3AS/314, MAAS/314

MAT3

MATHEMATICAL TRIPOS Part III

Friday, 2 June, 2023 1:30 pm to 4:30 pm

PAPER 314

ASTROPHYSICAL FLUID DYNAMICS

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

Attempt no more than **THREE** questions.
There are **FOUR** questions in total.
The questions carry equal weight.

The paper will start with a reminder of key facts.
You may refer to these formulae in your solutions, but, please make sure to provide sufficient details when using them.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

You are reminded of the equations of ideal magnetohydrodynamics in the form

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0, \qquad \frac{\partial p}{\partial t} + \mathbf{u} \cdot \nabla p + \gamma p \nabla \cdot \mathbf{u} = 0, \tag{1}$$

$$\rho \left(\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} \right) = -\rho \nabla \Phi - \nabla p + \frac{1}{4\pi} (\nabla \times \mathbf{B}) \times \mathbf{B}, \tag{2}$$

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B}), \qquad \nabla^2 \Phi = 4\pi G \rho. \tag{3}$$

Conservation laws for momentum and energy

$$\frac{\partial (\rho \mathbf{u})}{\partial t} + \nabla \cdot \hat{\Pi} = 0, \quad \hat{\Pi}_{ij} = \rho u_i u_j + \left(p + \frac{B^2}{8\pi} \right) \delta_{ij} - \frac{B_i B_j}{4\pi}, \tag{4}$$

$$\frac{\partial}{\partial t} \left[\rho \left(\frac{u^2}{2} + e \right) + \frac{B^2}{8\pi} \right] + \nabla \cdot \left[\rho \mathbf{u} \left(\frac{u^2}{2} + h \right) + c \frac{\mathbf{E} \times \mathbf{B}}{4\pi} \right] = 0, \tag{5}$$

where h is the enthalpy obeying $\mathrm{d}h = T\ \mathrm{d}s + \rho^{-1} \mathrm{d}p$; $h = c_s^2/(\gamma - 1)$ for a polytropic gas with adiabatic index γ, where c_s is the speed of sound.

You may assume that for any scalar function f

$$\nabla f = \frac{\partial f}{\partial R} \mathbf{e}_R + \frac{1}{R} \frac{\partial f}{\partial \phi} \mathbf{e}_\phi + \frac{\partial f}{\partial z} \mathbf{e}_z \quad \textit{(cylindrical coordinates)} \tag{6}$$

$$\nabla f = \frac{\partial f}{\partial r} \mathbf{e}_r + \frac{1}{r} \frac{\partial f}{\partial \theta} \mathbf{e}_\theta + \frac{1}{r \sin\theta} \frac{\partial f}{\partial \phi} \mathbf{e}_\phi \quad \textit{(spherical coordinates).} \tag{7}$$

You may assume that for any vectors $\mathbf{C}$ *and* $\mathbf{D}$

$$(\nabla \times \mathbf{C}) \times \mathbf{C} = (\mathbf{C} \cdot \nabla) \mathbf{C} - \frac{1}{2} \nabla \left(|\mathbf{C}|^2 \right), \tag{8}$$

$$\nabla \times (\mathbf{C} \times \mathbf{D}) = \mathbf{C} (\nabla \cdot \mathbf{D}) + (\mathbf{D} \cdot \nabla) \mathbf{C} - \mathbf{D} (\nabla \cdot \mathbf{C}) - (\mathbf{C} \cdot \nabla) \mathbf{D}, \tag{9}$$

$$\nabla \cdot (\mathbf{C} \times \mathbf{D}) = \mathbf{D} \cdot (\nabla \times \mathbf{C}) - \mathbf{C} \cdot (\nabla \times \mathbf{D}). \tag{10}$$

and in cylindrical coordinates

$$\nabla \cdot \mathbf{C} = \frac{1}{R} \frac{\partial (R C_R)}{\partial R} + \frac{1}{R} \frac{\partial C_\phi}{\partial \phi} + \frac{\partial C_z}{\partial z}, \tag{11}$$

$$\nabla \times \mathbf{C} = \left(\frac{1}{R} \frac{\partial C_z}{\partial \phi} - \frac{\partial C_\phi}{\partial z} \right) \mathbf{e}_R + \left(\frac{\partial C_R}{\partial z} - \frac{\partial C_z}{\partial R} \right) \mathbf{e}_\phi + \frac{1}{R} \left[\frac{\partial (R C_\phi)}{\partial R} - \frac{\partial C_R}{\partial \phi} \right] \mathbf{e}_z, \tag{12}$$

$$(\mathbf{C} \cdot \nabla) \mathbf{C} = \left[(\mathbf{C} \cdot \nabla) C_R - \frac{C_\phi^2}{R} \right] \mathbf{e}_R + \left[(\mathbf{C} \cdot \nabla) C_\phi + \frac{C_R C_\phi}{R} \right] \mathbf{e}_\phi + \left[(\mathbf{C} \cdot \nabla) C_z \right] \mathbf{e}_z. \tag{13}$$

Equation $y'' + x^{-1} y' + y = 0$ has a solution $y(x) = y_0 J_0(x)$, where $J_0(x)$ is a Bessel function of order 0 and $y_0 = y(0)$ is a constant. Function $J_0(x)$ has infinite number of positive roots, the smallest of them is $x_1 \approx 2.4$, at which $J_0'(x_1) \approx -0.52$; also $J_0(0) = 1$, $J_0'(0) = 0$.

You may refer to these formulae in your solutions, but, please, make sure to provide sufficient details when using them.

1

A radio pulsar is a rapidly spinning neutron star (of radius $R_\star$) possessing a strong magnetic field. Let the neutron star spin vector be $\mathbf{\Omega}$. Its magnetic field $\mathbf{B}_0(\mathbf{r})$ is produced by currents running only inside the star ($r < R_\star$). Outside the neutron star there is plasma that rotates with the same angular speed $\mathbf{\Omega}$ as the star and forms the co-rotating *magnetosphere* of the radio pulsar.

(a) Working under the assumption of ideal, non-relativistic MHD and assuming $\mathbf{B}_0(\mathbf{r})$ to be the only magnetic field present, determine the spatial density of electric charges $n_q(\mathbf{r})$ outside the neutron star in the inertial frame (the final expression for $n_q(\mathbf{r})$ should contain no differential operators).

(b) Assume now that a neutron star can be modelled as an *aligned dipole*, i.e. its magnetic field for $r > R_\star$ is

$$\mathbf{B}_0(\mathbf{r}) = 3\frac{(\boldsymbol{\mu}\cdot\mathbf{r})\mathbf{r}}{r^5} - \frac{\boldsymbol{\mu}}{r^3},$$

with the magnetic moment $\boldsymbol{\mu}$ parallel to $\mathbf{\Omega}$.

(i) Assuming the charges outside the neutron star to be orbiting around it with the angular speed $\mathbf{\Omega}$ and adopting spherical coordinates (r, θ, ϕ) aligned with $\mathbf{\Omega}$, determine the corresponding current density as a function of r and θ for $r > R_\star$.

(ii) This current outside the neutron star generates its own (induced) magnetic field $\mathbf{B}_{\rm i}(\mathbf{r})$. Using dimensional arguments and focusing on the midplane ($\theta = \pi/2$) of the aligned dipole magnetosphere determine the characteristic distance R_e at which the strength of the induced field becomes comparable to the strength of the original field $\mathbf{B}_0(\mathbf{r})$. Provide a physical interpretation of R_e and argue that the assumption of an ideal, non-relativistic MHD should become invalid at this distance.

(iii) Consider a field line of the original vacuum field $\mathbf{B}_0(\mathbf{r})$ that crosses the midplane ($\theta = \pi/2$) at the radius R_e. Determine the co-latitude $\theta_\star$ of this field line at the neutron star surface (i.e. at $r = R_\star$) as one follows this field line back towards the origin.

2

A region in the Galaxy contains gas with uniform density ρ_0 and uniform magnetic field $\mathbf{B}_0$. Part of this region starts to collapse under its own gravity in an axisymmetric fashion in the direction perpendicular to $\mathbf{B}_0$, eventually forming a gaseous filament aligned with $\mathbf{B}_0$; the rest of the gas gets expelled from the vicinity of the filament. The filament can be considered as a static, non-rotating, axisymmetric and infinitely long structure, supported by the balance of its own gravity, pressure and magnetic stresses. The equation of state of the gas comprising the filament is $P = K\rho^2$, where P and ρ are gas pressure and density and K is a constant. Adopt cylindrical (R, ϕ, z) coordinates (aligned with $\mathbf{B}_0$) to describe the filament.

(a) Assume first that no toroidal magnetic field (in the ϕ-direction) gets generated in the process of collapse.

(i) Derive a single second-order differential equation describing the density distribution $\rho(R)$ inside the filament. Solve this equation and obtain $\rho(R)$, provided that the central density is $\rho(0) = \rho_1$.

(ii) Show that the filament has a finite radial extent and determine the radius of the filament (with all relevant constant factors).

(iii) Determine the mass per unit length (along $\mathbf{B}_0$) of the filament (with all relevant constant factors).

(b) Assume now that in the process of collapse an axisymmetric toroidal field B_ϕ (independent of z) gets generated in the filament, modifying its structure compared to part (a) such that the density distribution in the filament is given by

$$\rho(R) = \rho_1 e^{-R^2/L^2},$$

where ρ_1, L are constants. Determine the behavior of $B_\phi(R)$ assuming a magnetostatic equilibrium. By exploring the behavior of this solution for $R \to 0$ and $R \to \infty$, find the condition that L must satisfy for this solution to be meaningful.

3

Consider the following model of a steady, spherically-symmetric wind launched from the surface of a star with mass $M_\star$ and radius $R_\star$. Gas is heated and rapidly accelerated at the stellar surface so that at $r = R_\star$ its sound speed is $c_{s,0}$ and radial velocity is $u_0 \leqslant c_{s,0}$. This drives the gas outflow from the potential well of the star. The outflow undergoes a smooth transonic transition at the sonic radius r_s and expands to infinity. The gas is isentropic with adiabatic index $\gamma > 1$.

(a) Derive the equation setting the condition for a smooth passage of the flow through the sonic point and state this condition.

(b) Let us introduce $\alpha = c_{s,0}^2/\left(GM_\star/R_\star\right)$ and $\mathcal{M}_0 = u_0/c_{s,0}$, the Mach number of the flow at $r = R_\star$. Derive the equation that relates $\mathcal{M}_0$ and α demonstrating that for a transonic flow u_0 and $\mathcal{M}_0(\alpha, \gamma)$ are not arbitrary but are set by α. Determine the range of γ for which the transonic flow is possible, in principle.

(c) Use the equation derived in part (b) to show that for $\alpha = 1/2$ the flow admits a marginally transonic solution, i.e. a solution with a sonic point at $r = R_\star$. How does this result depend on γ? Are there other values of α for which $\mathcal{M}_0 = 1$ is possible?

(d) Use the equation derived in part (b) to show that for a transonic solution there are two values of α that correspond to any given $\mathcal{M}_0 < 1$. Determine the behavior of $\mathcal{M}_0(\alpha, \gamma)$ in the limit of a very hot outflow, when $\alpha \to \infty$.

4

A supernova explosion drives a strong, spherically symmetric shock wave into the surrounding interstellar medium. Because of the cooling processes operating inside the shocked medium, the radius of the shock $R(t)$ evolves as

$$R(t) = Ct^{1/3},$$

while the velocity inside the shock behaves as

$$u(R, t) = u_{\rm ps}(t)\,\frac{r}{R(t)},$$

where $u_{\rm ps}(t)$ is the post-shock gas velocity, i.e. $u_{\rm ps}(t) = u(r \to R(t), t)$. The gas into which the shock propagates can be considered as cold and having a constant density ρ_0; the adiabatic index of the gas is γ.

(a) Derive the post-shock values of the gas density $\rho_{\rm ps}(t) = \rho(r \to R(t), t)$, pressure $p_{\rm ps}(t) = p(r \to R(t), t)$, and velocity $u_{\rm ps}(t) = u(r \to R(t), t)$, assuming the shock to be strong and passage of the gas through the shock to be adiabatic.

(b) Determine the pressure at the centre of the sphere $p(0, t)$ for $\gamma = 5/3$.

(c) Suppose now that the medium into which the shock runs is pervaded by a uniform magnetic field $\mathbf{B}_0$, which is too weak to affect the force balance and the fluid motion. Working in a spherical coordinate system (r, θ, ϕ) aligned with $\mathbf{B}_0$, write down the post-shock magnetic field $\mathbf{B}_{\rm ps}(t) = \mathbf{B}(r \to R(t), t)$ in spherical coordinates for arbitrary γ.

(d) Determine the behaviour of the post-shock Alfven speed $v_A(r \to R(t), t)/v_{A,0}$ as a function of spherical coordinates (for arbitrary γ), where $v_{A,0}$ is the Alfven speed in the ambient medium. Find the highest value of this ratio.

END OF PAPER