UNIVERSITY OF CAMBRIDGE

MAMA/313, NST3AS/313, MAAS/313

MAT3
MATHEMATICAL TRIPOS Part III

Tuesday, 6 June, 2023 9:00 am to 11:00 am

PAPER 313

SOLITONS, INSTANTONS AND GEOMETRY

Before you begin please read these instructions carefully

Candidates have TWO HOURS to complete the written examination.

Attempt no more than **TWO** questions.
There are **THREE** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Let $\phi : \mathbb{R}^{D+1} \to \mathbb{R}$. Define the term soliton in the context of scalar Lagrangian field theory. Consider the Lagrangian

$$L = \int_{\mathbb{R}^D} \Big(\frac{1}{2}(\partial_t \phi)^2 - \frac{1}{2}|\nabla\phi|^2 - \kappa^2 |\nabla\phi|^4 - U(\phi) \Big) d^D\mathbf{x}$$

where κ is a constant and $U(\phi) \geqslant 0$. Use the Derrick scaling argument to find the range of spatial dimensions D where solitons cannot exist.

Assume that $\kappa = 0, D = 1$ and

$$U(\phi) = \phi^6 - 8\phi^4 + 16\phi^2.$$

Sketch the potential and explain why there are no kink solutions connecting the vacua ϕ_- and ϕ_+ if $\phi_- < 0$ and $\phi_+ > 0$.

Find a kink solution, and compute its mass.

2 The Bogomolny equations for the Abelian Higgs model on $\mathbb{R}^2$ with the flat metric $g = dx^2 + dy^2$ are

$$B = \frac{1}{2}(1 - |\phi|^2), \quad (D_x + iD_y)\phi = 0, \tag{1}$$

where ϕ is a complex scalar field, B is the magnetic field of the Abelian gauge potential A, and $D = d - iA$.

Derive the Taubes equation for a scalar function u on $\mathbb{R}^2 \setminus \{0\}$ such that $|\phi|^2 = e^u$. State the boundary conditions corresponding to a vortex solution where ϕ has a single zero of multiplicity N at the origin of $\mathbb{R}^2$.

Use the maximum principle to show that $u \leqslant 0$ everywhere on $\mathbb{R}^2$.

Assume that $u = u(r)$ depends only on the radial coordinate and find the constants (α, γ, δ) such that near the origin u takes the form

$$u \sim \alpha \ln r + \beta + \gamma r + \delta r^2 + \ldots .$$

[*The constant β need not be determined.*]

Consider a conformally rescaled metric $\tilde{g} = e^u(dx^2 + dy^2)$ where u solves the Taubes equation on $(\mathbb{R}^2, g)$, and assume that $\tilde{u}$ is a solution of the Taubes equation on $(\mathbb{R}^2, \tilde{g})$ with the vortex number $\widetilde{N}$. Show that $u + \tilde{u}$ is a solution of the Taubes equation on $(\mathbb{R}^2, g)$ and find its vortex number.

[*You can assume that the Bogomolny equations hold on $(\mathbb{R}^2, \tilde{g})$ with $B = \partial_x A_y - \partial_y A_x$, and that the Laplacians of $\tilde{g}$ and g are related by $\Delta_{\tilde{g}} = e^{-u}\Delta_g$.*]

3 Let g be a Euclidean metric on $\mathbb{R}^4$, and let vol be a volume form. Define the Hodge $\star$–operator of $(\mathbb{R}^4, g, vol)$ and find an expression for $\star^2$ on $\Lambda^2(\mathbb{R}^4)$.

Define a decomposition of $\Lambda^2(\mathbb{R}^4)$ into self-dual (SD) and anti-self-dual (ASD) two–forms, and show that $H \wedge G = 0$ if H is SD and G is ASD. Use this, together with a Bogomolny argument, to show that the $SU(n)$ Yang–Mills action action on $\mathbb{R}^4$ is bounded from below by a multiple of the second Chern number.

Let

$$g = d\mathbf{x} \cdot d\mathbf{x} + d\tau^2$$

and let A be an $\mathfrak{su}(2)$–valued one–form on $\mathbb{R}^4$. Consider a gauge in which $A_\tau = 0$ vanishes to show that ASD Yang–Mills equations take the form

$$\frac{\partial A_i}{\partial \tau} = \frac{1}{2}\epsilon_{ijk} F_{jk}, \quad k = 1, 2, 3.$$

Find the Lax pair for the ASDYM in this gauge.

END OF PAPER