UNIVERSITY OF
CAMBRIDGE

MAMA/312, NST3AS/312, MAAS/312

MAT3
MATHEMATICAL TRIPOS Part III

Tuesday, 6 June, 2023 9:00 am to 12:00 pm

PAPER 312

FIELD THEORY IN COSMOLOGY

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

Attempt no more than **THREE** questions.
There are **FOUR** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS	**SPECIAL REQUIREMENTS**
Cover sheet	None
Treasury tag	
Script paper	
Rough paper	

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Here you will compute the trispectrum induced by

$$H_{int} = \int d^3x\, a^4 \lambda \epsilon_{ijk} \phi_1 \left(\partial_i \phi_2\right) \left(\partial_j \phi_3\right) \left(\partial_k \phi_4\right) \,,$$

where $\phi_{1,2,3,4}$ are four distinct massless scalars with de Sitter mode functions

$$f(k,\tau) = \frac{H}{\sqrt{2k^3}} \left(1 + ik\tau\right) e^{-ik\tau} \,.$$

(i) Under a parity transformation (point inversion) a scalar field transforms as

$$\phi(\mathbf{k}) \to \phi'(\mathbf{k}) = P\phi(\mathbf{k})P = \phi(-\mathbf{k}) \,,$$

with P the parity operator. Compute $PH_{int}P$ and use it to derive

$$\langle [H_{int}, \prod_a^n \phi_a(\mathbf{k}_a)] \rangle = 2\,\mathrm{Re} \langle H_{int} \prod_a^n \phi_a(\mathbf{k}_a) \rangle \,.$$

(ii) Then, use the in-in formalism to derive a time integral expression for $\langle \prod_a^n \phi_a(\mathbf{k}_a) \rangle$ to linear order in λ.

(iii) Via an appropriate contour rotation or otherwise, show that the following integral is purely imaginary for any integer $p \geqslant 0$,

$$\int_{-\infty(1-i\epsilon)}^{0} d\tau\, e^{-ik_T\tau} (i\tau)^p \,,$$

(iv) Hence, compute the final late time trispectrum using that

$$\int_{-\infty(1-i\epsilon)}^{\tau_0} \frac{d\tau}{\tau} e^{-ik_T\tau} \to \gamma_E + \ln(|k_T\tau_0|) - i\frac{\pi}{2} \quad \text{as} \quad \tau_0 \to 0 \,,$$

where γ_E is Euler's constant.

(v) Prove non-perturbatively at the level of correlators that the power spectrum and bispectrum of ϕ are always parity even. Moreover, show that any parity-odd interaction of three scalars in the action vanishes if fields vanish at spatial infinity.

2

Derive the soft-graviton theorem for the scalar-scalar-graviton bispectrum as follows.

(i) Let the metric be

$$ds^2 = a^2\left[-d\eta^2 + (\delta_{ij} + \gamma_{ij})dx^i dx^j\right] ,$$

where

$$\gamma_{ij}(\mathbf{x}) = \int_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}} \sum_s \epsilon^s_{ij}(\mathbf{k})\gamma^s(\mathbf{k}) , \qquad \gamma^s(\mathbf{k}) = a_{s,\mathbf{q}} f_q(\eta) + a^\dagger_{s,\mathbf{q}} f^*_q(\eta) ,$$

is the graviton field. Consider a large change of coordinates $\epsilon^\mu = \{0, \omega_{ij}x^j\}$ with ω_{ij} constant in time, symmetric and traceless. Working to first order in ω_{ij} and zeroth order in γ_{ij} and φ compute the resulting transformation

- $\Delta\gamma_{ij} = -\nabla_\mu\epsilon_\nu - \nabla_\nu\epsilon_\mu$ of the transverse-traceless graviton perturbations γ_{ij}, and
- $\Delta\varphi = -\epsilon^\mu\nabla_\mu\varphi$ of a canonical massless scalar with vanishing expectation value.

[*Hint: use a symmetry argument to show that the relevant Christoffel symbols vanish.*]

(ii) Define the following charge operator,

$$Q_S = -2\omega_{ij}\int d^3x\, \Pi_{ij}(x) ,$$

where Π_{ij} is the momentum conjugate of the graviton,

$$\Pi_{ij}(\mathbf{x},\eta) = \int_{\mathbf{q}} e^{i\mathbf{x}\cdot\mathbf{q}} \sum_s \epsilon^s_{ij}(\mathbf{q})\Pi^s(\mathbf{q}) , \qquad \Pi^s(\mathbf{q}) = a_{s,\mathbf{q}} g_q(\eta) + a^\dagger_{s,\mathbf{q}} g^*_q(\eta) .$$

Using that $2\sum_s \epsilon^s_{ij}(\mathbf{0})\epsilon^s_{mn}(-\mathbf{0}) \simeq \delta_{im}\delta_{jn} + \delta_{in}\delta_{jm} - \delta_{ij}\delta_{mn}$ check that Q_S induces the gauge transformation $\Delta\gamma_{ij}$ you computed in (i).

(iii) Consider the Ward-Takahashi identity

$$i\langle[Q_S, \varphi(\mathbf{k}_1)\varphi(\mathbf{k}_2)]\rangle = \langle\Delta\left(\varphi(\mathbf{k}_1)\varphi(\mathbf{k}_2)\right)\rangle . \qquad (1)$$

On the left-hand side, trade the commutator for an appropriate imaginary part and then, using that $\Pi_{ij}|0\rangle \propto \gamma_{ij}|0\rangle$, re-write the result so that it is proportional to the soft bispectrum $\langle\varphi(\mathbf{k}_1)\varphi(\mathbf{k}_2)\gamma_{ij}(\mathbf{0})\rangle$.

(iv) Compute the right-hand side of (1) and hence state the final result for the soft-graviton theorem.

3

(i) Derive the Boltzmann equation for freely propagating photons to linear order in temperature perturbations $\Theta(\eta, \mathbf{k}, \hat{p})$,

$$\frac{\partial \Theta}{\partial \eta} + i\,(\hat{p} \cdot \mathbf{k})\,\Theta - \frac{d \ln \epsilon}{d\eta} = 0\,. \tag{1}$$

(ii) Starting from Eq. (1) derive the continuity and Euler equations for the Legendre multipoles

$$\Theta_l(\eta, \mathbf{k}) \equiv i^l \int \frac{d\mu}{2}\, \mathcal{P}_l(\mu) \Theta(\eta, \mathbf{k}, \mu = \hat{k} \cdot \hat{p})\,. \tag{2}$$

[*Hint: You may use that* $\mathcal{P}_0(x) = 1$, $\mathcal{P}_1(x) = x$ *and* $\mathcal{P}_2(x) = (3x^2 - 1)/2$]

(iii) To first order in perturbations the geodesic equation gives

$$\frac{d \ln \epsilon}{d\eta} = -\frac{d\Psi}{d\eta} + \left(\Phi' + \Psi'\right)\,.$$

Assuming that the Newtonian potentials are homogeneous and equal to each other, but change from time η_i to time $\eta_f > \eta_i$, compute $\Theta(\eta_f, \mathbf{x}_0, \hat{p})$ from Eq. (1) using the line of sight solution in terms of an appropriate initial condition to be specified. Discuss what contributions to the CMB angular power spectrum are captured by this calculation.

4

Consider the equations of standard perturbation theory for collisionless dark matter,

$$\delta' + \nabla \cdot [(1+\delta)\,\mathbf{v}] = 0\,, \qquad v_i' + \mathcal{H} v_i + (\mathbf{v} \cdot \nabla)\, v_i = -\nabla_i \phi\,, \qquad \nabla^2 \phi = \frac{3}{2} \mathcal{H}^2 \Omega_m \delta\,. \tag{1}$$

where a prime denotes a derivative with respect to conformal time τ.

(i) Derive the linearized equation of motion for vorticity, $\mathbf{w} \equiv \nabla \times \mathbf{v}$, and determine its time dependence.

(ii) Derive the linearized coupled equations of motion for density δ and velocity divergence $\theta \equiv \nabla \cdot \mathbf{v}$. Then derive a single second-order differential equation for δ by eliminating θ. Assuming an Einstein-de Sitter universe with $\mathcal{H} = 2/\tau$ and $a = (\tau/\tau_0)^2$, state or derive the leading time dependence of δ and explicitly show that it is a solution. Hence determine the leading time dependence of θ.

(iii) Draw all necessary diagrams to compute the density four-point function at tree level and at one loop, namely up to $\mathcal{O}((\delta^{(1)})^8)$. Hence, write down expressions for the tree-level diagrams in terms of the kernels F_n. Finally, discuss whether one should include additional interactions beyond those appearing in (1) to obtain a consistent prediction for the one-loop trispectrum.

END OF PAPER