UNIVERSITY OF CAMBRIDGE

MAMA/311, NST3AS/311, MAAS/311

MAT3

MATHEMATICAL TRIPOS Part III

Friday, 9 June, 2023 1:30 pm to 4:30 pm

PAPER 311

BLACK HOLES

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

Attempt no more than **THREE** questions.
There are **FOUR** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 The following metric solves Einstein's equation in 3 dimensions with a negative cosmological constant:

$$\mathrm{d}s^2 = -f(r)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{f(r)} + r^2\left(\mathrm{d}\phi - \Omega(r)\,\mathrm{d}t\right)^2$$

where $\phi \sim \phi + 2\pi$,

$$f(r) = \frac{(r^2 - r_+^2)(r^2 - r_-^2)}{r^2 L^2}\,, \qquad \Omega(r) = \frac{r_+ r_-}{L\, r^2}\,,$$

and L, r_+ and r_- are constants with $L > 0$ and $r_+ \geqslant r_- \geqslant 0$. Assume that $r > r_+$ and that $\partial/\partial t$ is future-directed in (t, r, ϕ) coordinates.

(a) Show that

$$k = \frac{\partial}{\partial t} \qquad \text{and} \qquad m = \frac{\partial}{\partial \phi}$$

are Killing vector fields. [2]

(b) Construct the analogue of ingoing Kerr coordinates (v, r, χ) and determine the form of the metric in these coordinates. Show that this metric can be analytically extended across the surface $r = r_+$. [6]

(c) Show that the region $r_- < r < r_+$ is causally disconnected from the region $r > r_+$ in the ingoing Kerr coordinates of part (b). [6]

(d) Show that $r = r_+$ is a null hypersurface and is a Killing horizon of the Killing vector field $\xi = k + \Omega(r_+)\, m$ and determine its associated surface gravity. [6]

(e) Consider the case $r_- = 0$ and assume that there are no curvature singularities at $r = 0$.

(i) Show that the region $0 < r < r_+$ in Kerr coordinates (v, r, χ) contains trapped surfaces. [5]

(ii) How is the result obtained in part (i) consistent with Penrose's singularity theorem? *[Hint: look at the behaviour of null geodesics in (t, r, ϕ) coordinates near $r = 0$.]* [5]

2 Let $(\mathcal{M}, g)$ be a $D-$dimensional spacetime that is globally hyperbolic with a non-compact Cauchy surface Σ and $\mathcal{N}$ a null hypersurface in $(\mathcal{M}, g)$. Assume the spacetime satisfies the Einstein equation with matter obeying the null energy condition.

(a) Show that the normal to $\mathcal{N}$ is tangent to null geodesics within $\mathcal{N}$. [3]

(b) Define the expansion θ, shear $\hat{\sigma}_{ab}$ and rotation $\hat{\omega}_{ab}$ of a null geodesic congruence. [3]

(c) Derive Raychaudhuri's equation

$$\frac{\mathrm{d}\theta}{\mathrm{d}\lambda} = -\frac{1}{A}\theta^2 - \hat{\sigma}_{ab}\hat{\sigma}^{ab} + \hat{\omega}_{ab}\hat{\omega}^{ab} - R_{ab}U^aU^b\,,$$

for an affinely parametrised null congruence of geodesics with tangent vector U^a and affine parameter λ, where A is a constant you should determine. [8]

(d) Show that if $\theta = \theta_0 < 0$ at a point on a generator γ of a null hypersurface, then $\theta \to -\infty$ within finite affine parameter distance $A/|\theta_0|$ provided γ extends this far. You may assume that if the congruence contains the generators of a null hypersurface $\mathcal{N}$, then $\hat{\omega} = 0$ on $\mathcal{N}$. [4]

(e) Let $\mathcal{M}$ contain a trapped surface T . Let $\theta_0 < 0$ be the maximum value of θ on T for both sets of null geodesics orthogonal to T. Show that at least one of these geodesics is future-inextendible and has affine length no greater than $A/|\theta_0|$. You may assume that if S is a $(D-2)$-dimensional orientable spacelike submanifold of a globally hyperbolic spacetime, then every $p \in \dot{J}^+(S)$ lies on a future-directed null geodesic starting from S which is orthogonal to S and has no points conjugate to S between S and p. You may also assume that $\dot{J}^+(S)$ is an achronal submanifold and that p is conjugate to S if $\theta \to -\infty$ at p. [12]

3

(a) State and prove the version of the first law of black hole mechanics that relates the change in area of the event horizon to the energy and angular momentum of infalling matter. You may assume Raychaudhuri's equation, knowledge of Gaussian null coordinates and the zeroth law of black hole mechanics. [10]

(b) State the second law of black hole mechanics and sketch a proof. You may assume that the generators of the future horizon are complete to the future. [5]

(c) One suggested modification of general relativity involves a scalar Φ that couples to curvature via

$$G_{ab} = T_{ab}\,.$$

with

$$T_{ab} = \nabla_a\Phi\nabla_b\Phi - \frac{1}{2}g_{ab}\nabla_c\Phi\nabla^c\Phi + \xi[G_{ab}\Phi^2 - 2\nabla_a(\Phi\nabla_b\Phi) + 2g_{ab}\nabla^c(\Phi\nabla_c\Phi)]\,,$$

where ξ is a real constant and $G_{ab} = R_{ab} - Rg_{ab}/2$ is the Einstein tensor.

(i) Consider null geodesics $x^a = x^a(\lambda)$ with affine parameter λ and tangent vector $U^a = (\partial/\partial\lambda)^a$. Show that

$$\int_{-\infty}^{+\infty} T_{ab}U^aU^b\,\mathrm{d}\lambda \geqslant 0 \qquad (\star)$$

provided that $\xi < 0$ and $\Phi\Phi'/(1-\xi\Phi^2) \to 0$ as $|\lambda| \to +\infty$, where primes denote derivatives with respect to λ. [10]

(ii) Comment on the implications of $(\star)$ in part (i) to the second law of black hole mechanics. [5]

4

(a) A real scalar field ϕ is quantised in a globally hyperbolic background spacetime $(\mathcal{M}, g)$. In the remote past and the distant future, this metric is time independent. Describe in detail how to use the idea of Bogoliubov transformations to describe how the state of ϕ changes between the remote past and the distant future. [9]

(b) Suppose that in the remote past the system is in the vacuum state. Derive the formula, in terms of Bogoliubov coefficients, for the expectation value of the number of particles in a certain mode as measured by an observer in the far future, in the vacuum state as defined by an observer in the far past. [4]

(c) Sketch how considerations of this type lead to a picture of particle production by black holes. [2]

(d) Consider the Schwarzschild-AdS black hole in Eddington-Finkelstein coordinates (v, r, θ, ϕ)

$$\mathrm{d}s^2 = -f(r)\mathrm{d}v^2 + 2\,\mathrm{d}v\,\mathrm{d}r + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2)$$

with

$$f(r) = \frac{r^2}{L^2} + 1 - \frac{r_+}{r}\left(\frac{r_+^2}{L^2} + 1\right),$$

where r_+ and L are positive constants and $\theta \in [0, \pi]$ and $\phi \sim \phi + 2\pi$ are the usual spherical polar coordinates on the unit radius round two-sphere.

(i) Assuming $r = r_+$ is a Killing horizon of

$$k^a = \left(\frac{\partial}{\partial v}\right)^a,$$

determine the Hawking temperature T_H of the Schwarzschild-AdS black hole. [5]

(ii) Evaluate the generalised Komar energy

$$M = -\frac{1}{8\pi}\int \star\mathrm{d}(k - \bar{k})$$

where $\bar{k} = \lim_{r_+\to 0} k$ and where the integral is taken over a constant v, r surface and the orientation is $\mathrm{d}v \wedge \mathrm{d}r \wedge \mathrm{d}\theta \wedge \mathrm{d}\phi$. [5]

(iii) Verify the first law of black hole mechanics and find the range of values of r_+ for which the black hole is in stable equilibrium with an infinite reservoir of radiation at its Hawking temperature. [5]

END OF PAPER