UNIVERSITY OF CAMBRIDGE

MAMA/310, NST3AS/310, MAAS/310

MAT3

**MATHEMATICAL TRIPOS Part III**

Monday, 5 June, 2023 1:30 pm to 4:30 pm

# PAPER 310

# COSMOLOGY

**Before you begin please read these instructions carefully**

Candidates have THREE HOURS to complete the written examination.

Attempt no more than **THREE** questions.
There are **FOUR** questions in total.
The questions carry equal weight.

**STATIONERY REQUIREMENTS**
Cover sheet
Treasury tag
Script paper
Rough paper

**SPECIAL REQUIREMENTS**
None

**You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.**

**1**

**(a)** From the continuity equation, determine the scaling of the energy density $\rho$ with scale factor $a$ for a component with constant equation of state parameter $w$. Hence derive an expression for the Hubble parameter $H(z)$ in terms of $z$, $H_0$, and the density parameters $\Omega_{i,0}$ and equations of state $w_i$ of the different components $i$. Finally, show that the comoving distance along the line of sight to a certain redshift $z$ is given by

$$\chi(z) = \frac{1}{H_0} \int_0^z \frac{dz'}{\left[\sum_i \Omega_{i,0}(1+z')^{3(1+w_i)}\right]^{1/2}} \tag{1}$$

**(b)** The angular diameter distance $d_A$ relates the physical size perpendicular to the line of sight, $x$, of a small object to its observed angular size $\theta$ via $\theta = x/d_A$. For a flat universe $d_A$ is given by:

$$d_A(z) = \frac{\chi(z)}{1+z} \tag{2}$$

In all remaining parts of this question, consider a simple, flat LCDM cosmology with only two components: matter and standard dark energy (with $w = -1$). By considering both low- and high redshift limits of $d_A(z)$ or otherwise, show that the angular diameter distance – redshift relation must turn over, i.e., there is a redshift beyond which an object with fixed physical size appears larger in the sky if it is at higher redshift. Does such a turn-over also exist in cosmologies with non-standard dark energy, i.e. where the equation of state of the dark energy component deviates slightly from $w = -1$?

**(c)** Galaxy surveys are able to make precise observations of the angles $\theta(z)$ subtended at different redshifts by the "BAO scale" $\chi_d$, which is a standard comoving distance, here assumed oriented perpendicular to the line of sight. Assume that, while the value of $\chi_d$ is not known, it is fixed and does not evolve with redshift. Explain how such measurements of $\theta(z)$ can be used to determine the parameter $\Omega_{m,0}$ in the simple LCDM cosmology, stating how many different redshifts need to be probed to determine this parameter.

[*Hint: note that the angle subtended by* $\chi_d$ *is given by* $\theta = a\chi_d/d_A$.]

**(d)** Show that the comoving distance $\Delta\chi$ between two sources separated by a small redshift difference $\Delta z$ and by a small angle $\Delta\theta$ is given in the simple LCDM cosmology by

$$\Delta\chi(\Delta z, \Delta\theta) = \frac{f(z, \Omega_{m,0})}{H_0} \sqrt{\Delta z^2 + y^2(z, \Omega_{m,0})(\Delta\theta)^2} \tag{3}$$

to leading order in $\Delta z$ and $\Delta\theta$, where $f(z, \Omega_{m,0}) = \frac{1}{\sqrt{\Omega_{m,0}(1+z)^3 + (1-\Omega_{m,0})}}$ and $y(z, \Omega_{m,0})$ is a function you should specify (but may leave in integral form.)

[*Hint: you may assume that the comoving distance* $\Delta\chi$ *can be written in terms of comoving distances along* ($\Delta\chi_\parallel$) *and perpendicular to the line of sight* ($\Delta\chi_\perp$) *as* $\Delta\chi^2 = \Delta\chi_\parallel^2 + \Delta\chi_\perp^2$.]

**(e)** Now note that the comoving BAO scale $\chi_d$ can also be measured in redshift difference (not just angle). Can measurements of both angular and redshift differences corresponding to the fixed BAO scale be used to determine $\Omega_{m,0}$ even if only measurements around a single redshift are available and the value of $\chi_d$ is unknown? Briefly justify your answer.

**2**

**(a)** Explain carefully why the temperature of the cosmic neutrino background, $T_\nu$, is related to the CMB photon temperature, $T$, by

$$\frac{T_\nu}{T} = \left(\frac{4}{11}\right)^{1/3}. \tag{1}$$

In your explanation, you may assume without proof that the entropy $S = \frac{\rho+P}{T}V$ is conserved in an expanding universe.

**(b)** After neutrino decoupling, the relic neutrinos retain the relativistic Fermi-Dirac distribution function, which is given by

$$f(p) = \frac{1}{e^{(p-\mu_\nu)/T_\nu} + 1} \tag{2}$$

in terms of the magnitude of the momentum $p$ and the chemical potential of the neutrinos $\mu_\nu$.

First consider the standard case where the chemical potential is negligible. Write down an integral expression for the energy density of one relic neutrino species and show that in the limit of early times when $T_\nu \gg m_\nu$, with $m_\nu$ the small mass of this neutrino species, the energy density is:

$$\rho_\nu \approx \frac{7\pi^2}{120}(T_\nu)^4. \tag{3}$$

[*Hint: you may assume that* $g = 2$ *for this neutrino species and that the density of particles in phase space is* $\frac{g}{(2\pi)^3}f(p)$. *You may also use the standard integrals:* $\int_0^\infty \frac{x^c}{e^x+1} = c!\left(1 - \frac{1}{2^c}\right)\zeta(c+1)$, *where* $\zeta(2) = \frac{\pi^2}{6}$, $\zeta(3) \approx 1.202$, $\zeta(4) = \frac{\pi^4}{90}$, $\zeta(5) \approx 1.037$.]

**(c)** In the remainder of the question, consider a so-called "degenerate" neutrino which has a large and positive chemical potential $\mu_\nu \gg T_\nu$, a negligible neutrino mass, and $g = 1$. Show that the energy density of such a degenerate neutrino species is approximately given by

$$\rho_\nu \approx A\mu_\nu^4 \tag{4}$$

where you should specify the constant $A$. How does the energy density of the corresponding antineutrinos compare to this result? You may assume thermal equilibrium holds.

[*Hint: to perform the first phase space integral, you may wish to make a simple, constant approximation to* $f(p)$ *in the integrand in the regimes* $p < \mu_\nu$ *and* $p > \mu_\nu$ *and neglect the thin transition between these regimes.*]

**(d)** Derive an upper bound on $\mu_\nu/T_\nu$ by requiring that the energy density in degenerate neutrinos does not exceed the critical density $\rho_c$ today (which you may assume is given in terms of the CMB temperature by $\rho_c = 6 \times 10^4 T^4$).

**(e)** Which cosmological observations could distinguish a universe with degenerate neutrinos from the standard scenario? Briefly justify your answer.

**3** In this question you will discuss features in the CMB and matter power spectra. You may quote without proof results for the evolution of the potential perturbation $\Phi$ in different epochs.

**(a)** Starting from the continuity equation

$$\delta_r' + 3\mathcal{H}\left(\frac{\delta P_r}{\delta \rho_r} - \frac{\bar{P}_r}{\bar{\rho}_r}\right)\delta_r = -\left(1 + \frac{\bar{P}_r}{\bar{\rho}_r}\right)(\nabla \cdot \mathbf{v}_r - 3\Phi') \tag{1}$$

and the Euler equation

$$\mathbf{v}_r' + 3\mathcal{H}\left(\frac{1}{3} - \frac{\bar{P}_r}{\bar{\rho}_r}\right)\mathbf{v}_r = -\frac{\nabla \delta P_r}{\bar{\rho}_r + \bar{P}_r} - \nabla\Phi \tag{2}$$

(where $\delta_r, \mathbf{v}_r, \rho_r, P_r$ are the radiation density contrast, velocity perturbation, energy density, and pressure, primes indicate derivatives with respect to conformal time $\tau$, and bars indicate background quantities), show that the radiation perturbations during matter domination are described by the following differential equation:

$$\delta_r'' - \frac{1}{3}\nabla^2\delta_r = \frac{4}{3}\nabla^2\Phi. \tag{3}$$

**(b)** Determine the time-evolution during matter domination of the Sachs-Wolfe term $S(\mathbf{k},\tau) \equiv \frac{\delta_r}{4}(\mathbf{k},\tau) + \Phi(\mathbf{k},\tau)$, which is typically the largest contribution to the CMB temperature anisotropies. Hence sketch the shape of the Sachs-Wolfe transfer function $T_S$ on large scales as a function of wavenumber $k$, where $T_S$ is defined via the primordial comoving curvature perturbation $\mathcal{R}(\mathbf{k},0)$ by $S(\mathbf{k},\tau) = T_S(k,\tau)\mathcal{R}(\mathbf{k},0)$. Comment briefly on the relation of your result to the shape of the CMB power spectrum. Finally, explain briefly whether you would expect the baryon transfer function $T_b(k,\tau) \equiv \frac{\delta_b(\mathbf{k},\tau)}{\mathcal{R}(\mathbf{k},0)}$ (where $\delta_b$ is the baryon density contrast) to have oscillatory features soon after recombination.

**(c)** Sketch the transfer function for the cold dark matter comoving gauge density contrast, defined as $T_c(k,\tau) \equiv \frac{\Delta_c(\mathbf{k},\tau)}{\mathcal{R}(\mathbf{k},0)}$, soon after recombination. You may neglect any impact of baryons here (so that this is just a sketch of the matter transfer function.) Briefly explain the origin of the key features apparent in this transfer function.

**(d)** Finally, consider the growth of matter density perturbations after recombination, now including small corrections due to the baryons. The evolution of the cold dark matter (denoted $c$) and baryon (denoted $b$) densities, which together make up the total matter density ($m$), is described by two coupled differential equations

$$\ddot{\delta}_c + 2H\dot{\delta}_c - 4\pi G(\bar{\rho}_c\delta_c + \bar{\rho}_b\delta_b) = 0, \tag{4}$$

$$\ddot{\delta}_b + 2H\dot{\delta}_b - 4\pi G(\bar{\rho}_c\delta_c + \bar{\rho}_b\delta_b) = 0, \tag{5}$$

where $\delta$ is the relevant density contrast and dots indicate derivatives with respect to time $t$. From suitable combinations of these equations, determine uncoupled evolution equations for the variables $D \equiv \delta_c - \delta_b$ and $\delta_m = \frac{\bar{\rho}_c}{\bar{\rho}_m}\delta_c + \frac{\bar{\rho}_b}{\bar{\rho}_m}\delta_b$. Derive the evolution of both $D$ and $\delta_m$ with scale factor. Hence show that $\frac{\delta_c}{\delta_b} \to f$ at late times, where $f$ is a constant you should specify. Briefly comment on the physical interpretation of your result.

[*Hint: it may be helpful to show that* $\frac{\delta_c}{\delta_b} = \frac{\bar{\rho}_m\delta_m + \bar{\rho}_b D}{\bar{\rho}_m\delta_m - \bar{\rho}_c D}$.]

**(e)** Would you expect the matter power spectrum, when carefully including the impact of baryons, to have any oscillatory features? Very briefly motivate your answer using the results of previous sections of the question.

**4** In this question you will discuss single-field slow-roll inflation.

**(a)** A treatment of quantum fluctuations in single-field slow-roll inflation with a scalar field $\phi$ predicts the following dimensionless power spectrum of comoving curvature perturbations:

$$\Delta^2_{\mathcal{R}} = \left(\frac{H}{\dot{\phi}}\right)^2 \left(\frac{H}{2\pi}\right)^2, \tag{1}$$

where dots indicate time derivatives. Specify when the right hand side of this equation is to be evaluated; then show that the scalar spectral index $n_s \equiv 1 + \frac{d \ln \Delta^2_{\mathcal{R}}}{d \ln k}$ is given by

$$n_s - 1 = -6\epsilon_V + 2\eta_V, \tag{2}$$

where $\epsilon_V$ and $\eta_V$ are the first and second potential slow-roll parameters, respectively.

[*Hint: recall that* $\epsilon \equiv -\frac{d \ln H}{dN} = -\frac{\dot{H}}{H^2} = \frac{\frac{1}{2M^2_{\rm pl}}\dot{\phi}^2}{H^2}$ *is the first Hubble slow-roll parameter and* $\eta = \frac{d \ln \epsilon}{dN}$ *is the second Hubble slow roll parameter; you may assume that for the potential slow-roll parameters* $\epsilon_V, \eta_V$ *the following holds during slow-roll inflation:* $\epsilon_V \equiv \frac{M^2_{\rm pl}}{2}\left(\frac{V_{,\phi}}{V}\right)^2 = \epsilon$ *and* $\eta_V \equiv M^2_{\rm pl}\left(\frac{V_{,\phi\phi}}{V}\right) = 2\epsilon - \frac{\eta}{2}$.]

**(b)** Let $\tilde{N}(\phi)$ be the number of e-folds of inflation remaining before inflation ends, given a field value $\phi$. Show that

$$\tilde{N}(\phi) = \int_{\phi}^{\phi_e} d\tilde{\phi} \frac{1}{M_{\rm pl}\sqrt{2\epsilon_V(\tilde{\phi})}}, \tag{3}$$

where $\phi_e$ is the field value when inflation ends.

**(c)** Assume that the power spectrum of tensor modes arising from inflation is given by $\Delta^2_h(k) = \frac{8}{M^2_{pl}}\left(\frac{H}{2\pi}\right)^2$. Show that the tensor-to-scalar-ratio $r \equiv \Delta^2_h/\Delta^2_{\mathcal{R}}$ is given by:

$$r = C\epsilon_V, \tag{4}$$

where you should specify the constant $C$.

**(d)** Consider an inflation model with the potential $V(\phi) = V_0 - \frac{m^2\phi^2}{2}$ and with $\phi$ rolling towards $\phi > 0$ from $\phi = 0$. Assuming that for nearly all of the evolution of $\phi$, $V_0 \gg \frac{m^2\phi^2}{2}$, derive $\tilde{N}(\phi)$ and hence show that the tensor-to-scalar-ratio is given by:

$$r \approx fC(n_s - 1)^2 \left(\frac{\phi_e}{M_{pl}}\right)^2 e^{(n_s-1)\tilde{N}(\phi)} \tag{5}$$

where you should specify the constant $f$.

Derive an upper limit for $r$ in this model, assuming that $\phi_e < M_{pl}$, that $40 < \tilde{N} < 60$, and that $n_s - 1 = -0.04$.

## END OF PAPER