MAT3

MATHEMATICAL TRIPOS Part III

Friday, 2 June, 2023 9:00 am to 12:00 pm

PAPER 309

GENERAL RELATIVITY

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

Attempt no more than **THREE** questions.
There are **FOUR** questions in total.
The questions carry equal weight.

Throughout, units are chosen such that $c = G = 1$,
and the signature convention for spacetime is $(-+++)$.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Suppose (M, g) is a manifold of dimension $n > 2$ with metric g, and let ∇ be the Levi-Civita connection.

a) i) Define the Riemann tensor $R^a{}_{bcd}$. You should justify carefully why any expression you give defines a tensor.

ii) Show that in a coordinate basis

$$R^\tau{}_{\sigma\mu\nu} = \partial_\mu \Gamma_\sigma{}^\tau{}_\nu - \partial_\nu \Gamma_\sigma{}^\tau{}_\mu + \Gamma_\sigma{}^\rho{}_\nu \Gamma_\rho{}^\tau{}_\mu - \Gamma_\sigma{}^\rho{}_\mu \Gamma_\rho{}^\tau{}_\nu.$$

iii) Establish the Bianchi identities

$$R^a{}_{[bcd]} = 0, \qquad R^a{}_{b[cd;e]} = 0,$$

and the contracted Bianchi identity $R^a{}_{b;a} - \frac{1}{2} R_{;b} = 0$.
[You may assume the existence of normal coordinates about any point $p \in M$.]

b) We say that a metric has *isotropic curvature* if

$$R_{abcd} = K(g_{ac}g_{bd} - g_{ad}g_{bc}).$$

Show that K must be a constant, and relate it to the scalar curvature R.

c) The geodesic deviation equation is

$$T^a \nabla_a \left(T^b \nabla_b Y^c \right) = R^c{}_{dab} T^d T^a Y^b.$$

Explain briefly what the vectors T^a and Y^a represent.

d) Suppose now that g is a Lorentzian metric for a four-dimensional spacetime with isotropic curvature. Suppose an observer falls freely along the timelike curve $\lambda = \lambda(t)$, where t is proper time along the curve. The observer picks an orthonormal frame $\{e_\alpha(0)\}_{\alpha=0}^3$ at $\lambda(0)$ with $e_0(0) = \dot{\lambda}(0)$ and extends this to a local frame $\{e_\alpha(t)\}_{\alpha=0}^3$ along λ by parallel propagation.

i) Show that $\{e_\alpha(t)\}_{\alpha=0}^3$ remains orthonormal along λ.

ii) A second body falls freely along the curve $\tilde{\lambda} = \tilde{\lambda}(t)$, such that in a local coordinate system $\{x^\mu\}$ we may write

$$\tilde{\lambda}^\mu(t) = \lambda^\mu(t) + e^\mu_\alpha(t) y^\alpha(t),$$

where $e_\alpha(t) = e^\mu_\alpha(t) \frac{\partial}{\partial x^\mu}$. Suppose that initially $y^0(0) = \dot{y}^0(0) = 0$, $y^i(0) = Y^i$, $\dot{y}^i(0) = V^i$, $i = 1, 2, 3$, where Y^i and V^i are $O(\epsilon)$. Find $y^\alpha(t)$, distinguishing the cases $K < 0, K = 0, K > 0$, ignoring $O(\epsilon^2)$ corrections.

2 Here (M, g) is a Lorentzian manifold.

a) Under a small variation of the metric $g_{ab} \to g_{ab} + \delta g_{ab}$, derive formulae for the change to first order in *i*) the inverse metric; *ii*) the volume form

b) Consider the following action for a real scalar field ψ, where μ is a real constant:

$$S[\psi, g] = \frac{1}{2} \int_M \left(-g^{ab} \nabla_a \psi \nabla_b \psi - \mu^2 \psi^2 \right) d\mathrm{vol}_g$$

i) Show that under an arbitrary variation $\psi \to \psi + \delta\psi$, with $\delta\psi$ vanishing near ∂M, δS vanishes if and only if ψ solves the Klein–Gordon equation

$$\nabla_a \nabla^a \psi - \mu^2 \psi = 0.$$

ii) Find the energy-momentum tensor, T^{ab}, associated to this matter model, and show directly that $\nabla_a T^{ab} = 0$ when ψ satisfies the Klein–Gordon equation.

iii) State Killing's equation. Show that if K^a is a Killing vector, then $J^a = T^a{}_b K^b$ satisfies $\nabla_a J^a = 0$, provided that ψ satisfies the Klein–Gordon equation.

c) Consider $\mathbb{R}^4$ with coordinates $(t, \mathbf{x})$, where $\mathbf{x} = (x^1, x^2, x^3)$, equipped with the metric

$$g = -f^2(x) dt^2 + h_{ij}(x) dx^i dx^j.$$

Here $f > 0$ and h_{ij} is positive definite, with inverse h^{ij}.

i) Starting from the definition of the Lie derivative, explain why $\frac{\partial}{\partial t}$ is a Killing field for this metric.

ii) Suppose that ψ solves the Klein–Gordon equation on this background, and vanishes for large $|\mathbf{x}|$. By applying the divergence theorem on the region $[\tau_1, \tau_2] \times \mathbb{R}^3$ to the vector field J^a defined above, or otherwise, show that

$$E(\tau) = \frac{1}{2} \int_{\Sigma_\tau} \left[f^{-1} (\partial_t \psi)^2 + f h^{ij} \partial_i \psi \partial_j \psi + f \mu^2 \psi^2 \right] \sqrt{h} d^3 x$$

is independent of τ, where $h = \det h_{ij}$ and $\Sigma_\tau = \{(t, \mathbf{x}) \in \mathbb{R}^4 | t = \tau\}$

3

a) Suppose that a spacetime metric may be written in wave coordinates as a perturbation of the Minkoswki metric:

$$g_{\mu\nu} = \eta_{\mu\nu} + \epsilon h_{\mu\nu}, \qquad \eta_{\mu\nu} = \text{diag}(-1, 1, 1, 1).$$

Writing the energy-momentum tensor as $\epsilon T_{\mu\nu}$ and expanding to $O(\epsilon)$, derive the linearized Einstein equations in wave gauge

$$\partial^\rho \partial_\rho \overline{h}_{\sigma\nu} = -16\pi T_{\mu\nu}, \qquad \partial_\mu \overline{h}^\mu{}_\nu = 0,$$

where $\overline{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2} h_\tau{}^\tau \eta_{\mu\nu}$, and indices are raised and lowered with the Minkowski metric.

You may assume that in any coordinate basis the Ricci tensor may be written

$$R_{\sigma\nu} = -\frac{1}{2} g^{\mu\rho} \partial_\mu \partial_\rho g_{\sigma\nu} + \Gamma_{\lambda\tau\nu} \Gamma^{\lambda\tau}{}_\sigma + \Gamma_{\lambda\tau\nu} \Gamma^\tau{}_\sigma{}^\lambda + \Gamma_{\lambda\tau\sigma} \Gamma^\tau{}_\nu{}^\lambda$$
$$+ \frac{1}{2} \partial_\sigma \Gamma_{\mu\nu}{}^\mu + \frac{1}{2} \partial_\nu \Gamma_{\mu\sigma}{}^\mu - \Gamma_{\mu\lambda}{}^\mu \Gamma_\nu{}^\lambda{}_\sigma$$

and that the wave coordinate condition takes the form $\Gamma_\mu{}^{\nu\mu} = 0$.

b) In "almost inertial" coordinates (t, x, y, z) we model a star as a perfect fluid with $p = 0$, and mass-energy density ρ that is independent of t, with centre of mass at the origin. We assume ρ is small everywhere, and vanishes for $r^2 = |\mathbf{r}|^2 > R^2$, where $\mathbf{r} = (x, y, z)$. The star undergoes slow, rigid, rotation about the z-axis, so that the fluid four-velocity is given by $u^\mu = (1, -\Omega y, \Omega x, 0)$ and we neglect terms of $O(\Omega^2 R^2)$. The energy-momentum tensor is given by $T^{\mu\nu} = \rho u^\mu u^\nu$.

i) Show that conservation of the energy-momentum tensor implies that ρ is axisymmetric: $x\partial_y \rho - y\partial_x \rho = 0$.

ii) Show that $\overline{h}_{ij} = 0$ and

$$\overline{h}_{00}(\mathbf{r}) = 4 \int_{\mathbb{R}^3} \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d^3\mathbf{r}', \qquad \overline{h}_{0i}(\mathbf{r}) = 4\Omega \int_{\mathbb{R}^3} \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} (y', -x', 0) d^3\mathbf{r}'.$$

(You need not verify the gauge condition is satisfied).

iii) Ignoring terms of $O(mR^2/r^3)$, deduce that for $r \gg R$

$$g = -\left(1 - \frac{2m}{r}\right) dt^2 + \left(1 + \frac{2m}{r}\right)(dx^2 + dy^2 + dz^2) + \frac{4ma}{r^3}(ydx - xdy)dt$$

where $m = \int_{\mathbb{R}^3} \rho(\mathbf{r}') d^3\mathbf{r}'$, and you should give an expression for a in terms of ρ.

You may assume that the Poisson equation in three dimensions, $\nabla^2 \phi = f$, *has solution*

$$\phi(\boldsymbol{r}) = -\frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{f(\boldsymbol{r}')}{|\boldsymbol{r} - \boldsymbol{r}'|} d^3\boldsymbol{r}'$$

4 Here (M, g) is a four dimensional Lorentzian spacetime.

a) A perfect fluid is described by its four-velocity field, u^a, satisfying $u_a u^a = -1$, together with the pressure p and mass-energy density ρ. The associated energy-momentum tensor is given by

$$T_{ab} = (\rho + p)u_a u_b + p g_{ab}.$$

Show that conservation of T_{ab} implies the first law of thermodynamics and Euler's equation:

$$u^a \nabla_a \rho + (\rho + p)\nabla_a u^a = 0, \qquad (\rho + p)u^b \nabla_b u_a = -(g_{ab} + u_a u_b)\nabla^b p. \qquad (\dagger)$$

b) The three-sphere, S^3, can be parameterized by the Euler angles (θ, ϕ, ψ), where $0 < \theta < \pi$, $0 < \phi < 2\pi$, $0 < \psi < 4\pi$. Define the following 1-forms

$$\sigma^1 = -\sin\psi d\theta + \cos\psi \sin\theta d\phi, \quad \sigma^2 = \cos\psi d\theta + \sin\psi \sin\theta d\phi, \quad \sigma^3 = d\psi + \cos\theta d\phi.$$

Show that $d\sigma^i = \frac{1}{2}\delta^{il}\epsilon_{ljk}\sigma^j \wedge \sigma^k$, where i, j, k, l run over $1, 2, 3$ with summation convention assumed, and $\epsilon_{ijk} = \epsilon_{[ijk]}$ is the usual alternating tensor with $\epsilon_{123} = 1$.

c) Let $M = \mathbb{R} \times S^3$ be parameterised by (t, θ, ϕ, ψ), and consider the metric

$$g = -dt^2 + (\sigma^1)^2 + (\sigma^2)^2 + (\sigma^3)^2, \qquad (*)$$

with the σ^i as in part b) above. Let $e^0 = dt$ and $e^i = \sigma^i$, $i = 1, 2, 3$. Find the connection one-forms and curvature two-forms associated to the orthonormal frame $\{e^\mu\}_{\mu=0}^3$, and show that in this basis the only non-vanishing components of the Riemann tensor are

$$R_{ijkl} = \frac{1}{4}(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}).$$

Hence, find the Ricci and Einstein tensor for this spacetime.
You may assume without proof Cartan's first and second structure equations:

$$de^\mu + \omega^\mu{}_\nu \wedge e^\nu = 0, \qquad d\omega^\mu{}_\nu + \omega^\mu{}_\sigma \wedge \omega^\sigma{}_\nu = \Theta^\mu{}_\nu.$$

d) Deduce that the metric $(*)$ satisfies the Einstein equations with a cosmological constant Λ:

$$G_{ab} + \Lambda g_{ab} = 8\pi T_{ab}$$

for T_{ab} of the form $(\dagger)$ where $u = \frac{\partial}{\partial t}$, and ρ, p are constants which you should determine in terms of Λ. Show that your solution has vanishing pressure and positive mass-energy density for a particular choice of Λ, which you should state.

END OF PAPER