UNIVERSITY OF CAMBRIDGE

MAMA/307, NST3AS/307, MAAS/307

MAT3

MATHEMATICAL TRIPOS Part III

Friday, 9 June, 2023 9:00 am to 12:00 pm

PAPER 307

SUPERSYMMETRY AND DUALITY

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

Attempt no more than **THREE** questions.
There are **FOUR** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Briefly explain the meaning of the Kähler potential and superpotential in supersymmetric field theories.

A chiral superfield has expansion

$$\Phi(x,\theta,\bar{\theta}) = \phi(x) + \sqrt{2}\theta\psi(x) + \theta^2 F(x) + i\theta\sigma^\mu\bar{\theta}\partial_\mu\phi(x) - \frac{i}{\sqrt{2}}\theta^2\partial_\mu\psi(x)\,\sigma^\mu\bar{\theta} - \frac{1}{4}\theta^2\bar{\theta}^2\Box\phi(x)$$

Compute the bosonic terms in the action

$$\int d^4x \int d^4\theta \; \Phi^\dagger\Phi$$

Compute the bosonic terms in the action

$$\int d^4x \int d^2\theta \; W(\Phi)$$

Find the vacuum structure, including the vacuum energy, of theories with the following superpotentials:

i) Two chiral multiplets X and Z with $W = X^2 Z$.

ii) Three chiral multiplets X, Y and Z with $W = XYZ$.

iii) Three chiral multiplets X, Y and Z with $W = \alpha Y + \beta Y X^2 + \gamma XZ$ with $\alpha, \beta, \gamma \neq 0$ and $|\gamma|^2 \gg |\alpha\beta|$.

(You may assume a canonical Kähler potential for all fields.)

2 Supersymmetic QED consists of a $U(1)$ vector multiplet V coupled to N chiral multiplets Φ^i of charge $+1$ and N chiral multiplets $\tilde{\Phi}_i$ of charge -1, with $i = 1, \ldots, N$. A Fayet-Iliopoulos (FI) term ζ is added to the action. A neutral chiral multiplet X is coupled through the superpotential

$$W = \sum_{i=1}^{N} \tilde{\Phi}_i (X - m_i) \Phi_i$$

Write down the potential energy V.

For each of the following cases, describe the $V = 0$ ground states. If there is a moduli space of vacua, compute its dimension. If there are isolated vacua, compute how many:

i) $\zeta \neq 0$ and $m_i = 0$.

ii) $\zeta \neq 0$ and m_i are all distinct.

ii) $\zeta = 0$ and m_i are all distinct.

iii) $m_i = \zeta = 0$.

Consider the situation $N = 1$ and $\zeta = m_1 = 0$. Compute the Kähler potential and associated metric on the moduli space of vacua parameterised by $M = \tilde{\Phi}\Phi$.

What are the singularities of this metric and what is their physical meaning?

3 $SU(2)$ supersymmetric gauge theory is coupled to N_f chiral multiplets Φ_i in the fundamental representation and N_f chiral multiplets $\tilde{\Phi}_i$ in the anti-fundamental representation with $i = 1, \ldots, N_f$.

Classically, the theory has the following $U(1)$ symmetries

	$U(1)_B$	$U(1)_A$	$U(1)_R$
Φ	1	1	$1 - 2/N_f$
$\tilde{\Phi}$	-1	1	$1 - 2/N_f$

where $U(1)_R$ is an R-symmetry.

i) The coefficient of the one-loop beta function is $b_0 = 6 - N_f$. Determine the transformation of Λ^{b_0} under each of the $U(1)$ symmetries in the table, where Λ is the holomorphic strong coupling scale. [You may use the fact that $I(\text{adj}) = 4$ for $SU(2)$.]

ii) Write down the gauge invariant chiral fields for $N_f = 1$. What constraints do they obey? What superpotential, consistent with all symmetries, can form on the moduli space? What is the resulting supersymmetric ground state?

ii) Write down the gauge invariant chiral fields for $N_f = 2$. What classical constraints do they obey? What superpotential, consistent with all symmetries, can form on the moduli space? What quantum deformation of the classical constraints is consistent with the symmetries?

iv) Write down the gauge invariant chiral fields for $N_f = 3$. Show that the $U(1)_R$ and $U(1)_R^3$ 't Hooft anomalies can be matched by unconstrained mesons and baryons at the origin of moduli space.

4 Briefly explain what it means for two quantum field theories to be *Seiberg duals.*

Consider $SU(N)$ supersymmetric gauge theory coupled to a single chiral multiplet S in the symmetric ($\square\square$) representation and p chiral multiplets $\tilde{\Phi}$ in the anti-fundamental ($\overline{\square}$) representation.

i) For what value of p is the quantum theory consistent?

ii) Show that the quantum theory has an R-symmetry under which

$$R[S] = \frac{2-N}{2+N} \quad \text{and} \quad R[\tilde{\Phi}] = 1$$

iii) Show that the theory is asymptotically free for all N.

It is conjectured that this theory has a dual description given by Spin(8) supersymmetric gauge theory coupled to chiral multiplets with the following symmetries

	Spin(8)	$SU(p)$	$U(1)_R$
q	$\square$	$\overline{\square}$	1
t	**s**	**1**	-5
M	**1**	$\square\square$	0
U	**1**	**1**	12

where **s** is the spinor representation of Spin(8).

iv) Show that the R-symmetry is a good symmetry of the quantum theory.

v) What superpotential can you write down consistent with the symmetries?

vi) For what values of N is the theory infra-red free? What does the conjecture that the theories are Seiberg dual mean for the low-energy physics of the $SU(N)$ gauge theory?

vii) Compute the $SU(p)^3$ 't Hooft anomaly in both the $SU(N)$ theory and the Spin(8) theory.

[*The beta function for a supersymmetric gauge theory has one-loop coefficient*

$$b_0 = \frac{3}{2}I(\text{adj}) - \frac{1}{2}\sum_{\text{chirals}} I(R)$$

Spin(8) is the double cover of $SO(8)$. *You may use the following facts about the dimension, Dynkin index I and anomaly coefficient A of representations of* $SU(N)$ *and* Spin(8)*:*

$SU(N)$ Irrep	$\square$	adj	$\square\square$
dim	N	N^2-1	$\frac{1}{2}N(N+1)$
$I(R)$	1	$2N$	$N+2$
$A(R)$	1	0	$N+4$

Spin(8) Irrep	$\square$	**s**	adj
dim	8	8	28
$I(R)$	1	1	6
$A(R)$	0	0	0

together with $I(R) = I(\bar{R})$ *and* $A(R) = -A(\bar{R})$.]

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