UNIVERSITY OF CAMBRIDGE

MAMA/306, NST3AS/306, MAAS/306

MAT3

MATHEMATICAL TRIPOS Part III

Wednesday, 7 June, 2023 1:30 pm to 4:30 pm

PAPER 306

STRING THEORY

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

Attempt no more than **THREE** questions.
There are **FOUR** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Consider the action

$$S[X] = -\frac{1}{4\pi\alpha'} \int_{\mathbb{R}} d\tau \int_0^\pi d\sigma \left[\eta^{\alpha\beta}\, \partial_\alpha X^\mu \partial_\beta X^\nu \, \eta_{\mu\nu} \right]$$

for an open string where $\sigma^\alpha = (\tau, \sigma)$ are coordinates on the worldsheet with $\sigma \in [0, \pi]$, and where the field X describes a map to $\mathbb{R}^{1,25}$ with its flat Minkowski metric $\eta_{\mu\nu} = \mathrm{diag}(-, +, \ldots, +)$.

a) Derive the classical mode expansion of $X^\mu(\sigma, \tau)$, assuming Neumann boundary conditions $\partial_\sigma X^\mu|_{\sigma=0,\pi} = 0$. Write down the canonical commutation relations that the modes obey in covariant quantization.

b) Derive the Noether current $J^{\mu\nu}_\alpha$ associated to target space Lorentz transformations $X^\mu \mapsto \Lambda^\mu{}_\nu X^\nu$. Express the corresponding Noether charge $M^{\mu\nu}$ in terms of your modes.

c) Show that the commutation relations obeyed by the modes imply that the $M^{\mu\nu}$ indeed obey the standard Lorentz algebra

$$\left[M^{\mu\nu}, M^{\kappa\lambda} \right] = i \left(\eta^{\mu\kappa} M^{\nu\lambda} - \eta^{\nu\kappa} M^{\mu\lambda} + \eta^{\nu\lambda} M^{\mu\kappa} - \eta^{\mu\lambda} M^{\nu\kappa} \right) .$$

2

a) State the general form of the $T(z)T(w)$ OPE in a 2d CFT.

b) Consider the theory of a single scalar field X whose holomorphic stress tensor is

$$T(z) = -\frac{1}{\alpha'} : \partial X \, \partial X : (z) - q : \partial^2 X : (z) \, ,$$

where $\partial = \partial/\partial z$ and q is a constant. Assuming $X(z)X(w) \sim -\frac{\alpha'}{2} \ln(z-w)$, calculate the central charge of this theory.

c) Show that this stress tensor arises from the action

$$S[X] = \frac{1}{4\pi\alpha'} \int d^2\sigma \sqrt{g} g^{\alpha\beta} \partial_\alpha X \partial_\beta X + \frac{q}{4\pi} \int d^2\sigma \sqrt{g}\, X R^{(2)} \, ,$$

where $R^{(2)}$ is the Ricci scalar of the worldsheet metric. [*You may assume that* $\delta\left(\sqrt{g} R^{(2)}\right) = \sqrt{g} \nabla^\alpha \left(\nabla^\beta \delta g_{\alpha\beta} - g^{\gamma\delta} \nabla_\alpha \delta g_{\gamma\delta} \right)$ *under a small variation of the metric.*]

d) How may this theory be used to construct a bosonic string that has critical dimension $D < 26$? Comment on whether string perturbation theory would be reliable in your model.

3

a) What does it mean for a local operator $\mathcal{O}(w, \bar{w})$ in a two dimensional Euclidean CFT to be *primary*? What is meant by the *conformal weights* of a primary operator?

b) A theory of D free scalar fields $X^\mu(w, \bar{w})$ has stress tensor

$$T(z) = -\frac{\eta_{\mu\nu}}{\alpha'} : \partial X^\mu \partial X^\nu : (z)$$
$$\bar{T}(\bar{z}) = -\frac{\eta_{\mu\nu}}{\alpha'} : \bar{\partial} X^\mu \bar{\partial} X^\nu : (\bar{z})$$

where $\eta_{\mu\nu}$ is the Minkowski metric in the target space. Derive conditions on the constants $(\epsilon_{\mu\nu}, k_\mu)$ that are required for the normal-ordered operator

$$: \epsilon_{\mu\nu} \partial X^\mu \bar{\partial} X^\nu \, e^{ik \cdot X} :$$

to be primary, and find its conformal weights.

c) Write down the closed string vertex operators describing a graviton and the vertex operator describing a B-field in the target space. Explain the relevance of the conditions you found above. Further explain why your vertex operators are invariant under the gauge redundancies of the worldsheet theory. [*You should use the form without ghosts.*]

d) Show that the operator $\eta_{\mu\nu} : \partial X^\mu \bar{\partial} X^\nu \, e^{ik \cdot X} :$ is not primary for any $k_\mu \neq 0$, but that the modified operator

$$(\eta_{\mu\nu} + \xi_\mu k_\nu + k_\mu \xi_\nu) \; : \partial X^\mu \bar{\partial} X^\nu \, e^{ik \cdot X} :$$

is primary provided the constant vector ξ_μ obeys conditions that you should find.

4 The action for a closed bosonic string moving on a curved space-time with background metric $G_{\mu\nu}$ is

$$S_1[X] = \frac{1}{4\pi\alpha'} \int_\Sigma d^2\sigma \, G_{\mu\nu}(X) \partial^\alpha X^\mu \partial_\alpha X^\nu \,,$$

where we have gauge-fixed the worldsheet metric to be flat.

a) Without detailed calculation, explain why consistency of the string theory requires that the Ricci tensor of the background metric vanishes to leading order in α'.

b) Now suppose that the target space metric takes the form

$$G_{\mu\nu}(x) dx^\mu dx^\nu = G_{ij}(y) dy^i dy^j + V(y) dz^2$$

where $x^\mu = (y^i, z)$ for $i = 0, \ldots, 24$, with V and G_{ij} chosen so that indeed $R_{\mu\nu} = 0$. For this target metric, consider the new action

$$S_2[Y, Z, p] = \frac{1}{4\pi\alpha'} \int d^2\sigma \, G_{ij}(Y) \partial^\alpha Y^i \partial_\alpha Y^j + \pi\alpha' \int d^2\sigma \, V^{-1}(Y) p^\alpha p_\alpha + i \int d^2\sigma \, p^\alpha \partial_\alpha Z \,,$$

where $p_\alpha(\sigma)$ is a new worldsheet field. By carrying out the path integral over p_α, but neglecting any 1-loop deteminants, show that the worldsheet CFT associated to $S_2[Y, Z, p]$ is classically equivalent to the worldsheet CFT associated to $S_1[X]$.

c) Taking the worldsheet $\Sigma = \mathbb{R}^2$, carry out the Z path integral directly in the theory associated to $S_2[Y, Z, p]$. Hence show that the theory is classically equivalent to that of a string propagating on the background

$$\tilde{G}_{\mu\nu}(x) dx^\mu dx^\nu = G_{ij}(y) dy^i dy^j + \frac{1}{V(y)} d\tilde{z}^2$$

d) Briefly comment on the fact that $\tilde{G}_{\mu\nu}$ may not be Ricci flat even though $G_{\mu\nu}$ is. [*Hint: In fact, the 1-loop deteminants neglected above conspire to construct a dilaton background depending on $V(Y)$.*]

END OF PAPER