UNIVERSITY OF CAMBRIDGE

MAMA/305, NST3AS/305, MAAS/305

MAT3

# MATHEMATICAL TRIPOS Part III

Thursday, 8 June, 2023 9:00 am to 12:00 pm

## PAPER 305

## THE STANDARD MODEL

**Before you begin please read these instructions carefully**

Candidates have THREE HOURS to complete the written examination.

Attempt no more than **THREE** questions.
There are **FOUR** questions in total.
The questions carry equal weight.

**STATIONERY REQUIREMENTS**
Cover sheet
Treasury tag
Script paper
Rough paper

**SPECIAL REQUIREMENTS**
None

**You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.**

**1** Consider the algebra for the generators of the Poincaré group: $P^\mu, M^{\mu\nu}$

$$\begin{aligned}\left[P^\mu\ ,\ P^\nu\right] &= 0\\ \left[M^{\mu\nu}\ ,\ P^\sigma\right] &= i\left(P^\mu\,\eta^{\nu\sigma}\ -\ P^\nu\,\eta^{\mu\sigma}\right)\\ \left[M^{\mu\nu}\ ,\ M^{\rho\sigma}\right] &= i\left(M^{\mu\sigma}\,\eta^{\nu\rho}\ +\ M^{\nu\rho}\,\eta^{\mu\sigma}\ -\ M^{\mu\rho}\,\eta^{\nu\sigma}\ -\ M^{\nu\sigma}\,\eta^{\mu\rho}\right).\end{aligned}$$

Consider also the Pauli-Ljubanski vector $W_\mu := \frac{1}{2}\,\varepsilon_{\mu\nu\rho\sigma}\,P^\nu\,M^{\rho\sigma}$ satisfying

$$[W_\mu, P_\nu] = 0, \qquad [W_\mu, M_{\rho\sigma}] = i\,(\eta_{\mu\rho}W_\sigma - \eta_{\mu\sigma}W_\rho)\,, \qquad [W_\mu, W_\nu] = -i\,\varepsilon_{\mu\nu\rho\sigma}W^\rho P^\sigma\,.$$

(a) Show that the operators $C_1 = P^\mu P_\mu$ and $C_2 = W^\mu W_\mu$ commute with all the generators and are the Casimir operators of the Poincaré group.

(b) Consider the unitary representations of the Poincaré group corresponding to $C_1 = m^2 > 0$ and $C_1 = 0$. By picking 4-vectors in the forward light-cone, identify the Little group in each case and the corresponding finite dimensional unitary representations of the Little group that can be identified with massive and massless particles respectively.

(c) Define what is meant by helicity $\lambda$ for the $C_1 = 0$ case and show that $\lambda = 0, \pm 1/2, \pm 1, \pm 3/2, \pm 2, \cdots$.

(d) Consider a vector field:

$$A^\mu(x) = \sum_{\lambda=-1}^{1} \int \frac{d^3p}{16\pi^3 E_p}\left[\epsilon^\mu(p,\lambda)\,a_{p\lambda}\,e^{ipx} + h.c.\right] \tag{1}$$

Impose constraints on the polarisation vectors $\epsilon^\mu$ that reduce the 4 degrees of freedom corresponding to a vector in 4 dimensions to only 2 degrees of freedom required to describe a massless particle of helicity $\lambda = \pm 1$. Explain what is meant by the statement that gauge invariance is only a redundancy. Show then that for a given transition amplitude described by $\mathcal{M} = \epsilon^\mu \mathcal{M}_\mu$, Lorentz invariance requires $q^\mu \mathcal{M}_\mu = 0$ where $q^\mu$ corresponds to the momentum of the massless helicity $\lambda = \pm 1$ particle in the amplitude.

(e) Explain *in detail* how if $A_\mu$ corresponds to the photon field, the condition $q^\mu \mathcal{M}_\mu = 0$ can be used to prove charge conservation. What can be said for helicities $\lambda \geqslant 2$?

**2** Consider the bosonic sector of the electroweak theory with gauge symmetry $SU(2)_L \times U(1)_Y$ and corresponding gauge fields $W_\mu^a, a = 1,2,3$ and $B_\mu$. The symmetry is broken to $U(1)_{EM}$ by a complex scalar field $H$ which is a doublet of $SU(2)_L$ with hypercharge $Y = 1/2$.

(a) Write down the most general bosonic renormalisable Lagrangian consistent with locality, stability and the gauge symmetries for the fields $W_\mu^a, B_\mu$ and $H$. Show that the Lagrangian is unique up to four arbitrary parameters that you should identify and that, depending on the sign of the coefficient of $|H|^2$ in the scalar potential $V(H)$, symmetry breaking occurs.

(b) Select a particular value of $\langle H \rangle = (0, v/\sqrt{2})^T$ and expand around it as:

$$H = \frac{1}{\sqrt{2}}\, e^{-i\xi^a(x)T^a} \begin{pmatrix} 0 \\ v + h(x) \end{pmatrix} \tag{1}$$

with $T^a$ the $SU(2)_L$ generators, $\xi^a(x)$ the Goldstone modes and $h(x)$ the physical Higgs field. Show explicitly that the fields $\xi^a(x)$ appear in the Lagrangian only in the combination $\partial_\mu \xi^a T^a + g W_\mu^a T^a + \frac{1}{2} g' B_\mu$ and can then be eliminated by a gauge choice.

(c) Write down the quadratic part of the bosonic Lagrangian by diagonalising the mass matrix for the gauge fields. Write explicitly an expression for the mass of each of the particles. Explain why the massless state can be identified as the photon.

(d) Now, introduce three generations of quark fields $Q_L^i, u_R^i, d_R^i$, with $i = 1,2,3$ the generation index, transforming as $(\mathbf{3},\mathbf{2})_{Y=1/6}$; $(\bar{\mathbf{3}},\mathbf{1})_{Y=2/3}$ and $(\bar{\mathbf{3}},\mathbf{1})_{Y=-1/3}$ under $SU(3)_c \times SU(2)_L \times U(1)_Y$, respectively. Write down their gauge invariant couplings to the Higgs field $H$. Show that after expanding around the vacuum, the fact that $\langle H \rangle \neq 0$ gives rise to mass terms for the quark fields.

(e) Diagonalise the corresponding mass matrix. Then, after properly redefining the quark fields, show that the coupling of up and down quarks to $W$ bosons is not flavour diagonal and is determined by a unitary matrix $V = V_{CKM}$.

$$\mathcal{L}_{W-quarks} = \frac{g}{\sqrt{2}} \left[ W_\mu^+ \bar{u}_L^i \gamma^\mu V_{ij} d_L^j + W_\mu^- \bar{d}_L^i \gamma^\mu V_{ij}^\dagger u_L^j \right] \tag{2}$$

Count the number of free parameters in $V_{CKM}$.

(f) Complete the fermion spectrum of the Standard Model with the lepton fields $L_L^i, e_R^i$ $i = 1,2,3$, transforming as $(\mathbf{1},\mathbf{2})_{Y=-1/2}$; $(\mathbf{1},\mathbf{1})_{Y=-1}$ under $SU(3)_c \times SU(2)_L \times U(1)_Y$. Briefly explain how neutrinos can get a mass if right handed neutrinos are added to the spectrum transforming as $\nu_R = (\bar{\mathbf{1}},\mathbf{1})_{Y=0}$.

**3** Consider a complex scalar field $\Phi(x^\mu)$ and a Dirac fermion field $\Psi(x^\mu)$ with $x^\mu$ being the four-dimensional spacetime coordinates.

(a) Argue why these fields have scaling dimension $[\Phi] = 1$ and $[\Psi] = 3/2$ respectively. Write down the most general renormalisable Lagrangian density for these fields $\mathcal{L}(\Phi, \Psi, \partial_\mu \Phi, \partial_\mu \Psi)$, which is invariant under the global $U(1)$ transformation $\Phi \to e^{2i\alpha}\Phi$, $\Psi \to e^{i\alpha}\Psi$ with $0 \leqslant \alpha < 2\pi$.

(b) Explain under which conditions the global $U(1)$ symmetry is spontaneously broken. What is the unbroken subgroup?

(c) Compute the conserved Noether current $J^\mu$ and the corresponding charge $Q$.

(d) Explain why the naive prescription to couple $\Phi$ to a gauge field $A_\mu(x^\nu)$: $\mathcal{L}(\Phi, A_\mu)$ through the coupling $J^\mu A_\mu$ *does not* provide a gauge invariant coupling and determine the extra term that needs to be added in order to have a gauge invariant coupling. Does the result agree with the prescription of substituting partial derivatives with covariant derivatives? Explain why this issue does not appear for the case of the Dirac fermion $\Psi(x^\mu)$ coupled to a gauge field: $\mathcal{L}(\Psi, A_\mu)$.

(e) Write down a $U(1)$ invariant coupling between $\Phi$ and $\Psi$ that contributes to the mass of $\Psi$ if the symmetry is broken. What is the dimension of the operator? Describe the relevance of this term as compared to the original mass term for $\Psi$.

**4**

(a) The scale dependence of a renormalised coupling $g(\mu)$ is given, to leading order, by the $\beta$ function equation:

$$\mu \frac{d\alpha}{d\mu} = -\frac{b}{2\pi}\alpha^2,$$

where $\mu$ is the renormalisation scale and $b$ is a real constant. Consider an $SU(N_c)$ gauge theory with $N_f$ flavours in the fundamental representation of $SU(N_c)$ for which $b = \frac{1}{3}(11N_c - 2N_f)$. For QCD $N_c = 3$ and the gauge coupling can be called $g_3$.

(i) Solve the $\beta$ function equation for $\alpha_3 = g_3^2/(4\pi)$. Sketch the profile of $\alpha_3(\mu)$ and estimate the scale $\Lambda_{QCD}$, at which QCD becomes strongly coupled, as a function of $b$ and the renormalisation scale $\mu$.

(ii) Briefly explain the difference between theories in which $b < 0$ and $b > 0$.

(iii) Suppose there is an energy scale $M_{GUT} > M_Z$, where $M_Z$ is the mass of the $Z$ boson, at which the three gauge couplings of the Standard Model coincide $\alpha_1(M_{GUT}) = \alpha_2(M_{GUT}) = \alpha_3(M_{GUT})$ (corresponding to $U(1)$, $SU(2)_L$ and $SU(3)_c$, respectively). Show that this implies that at the $M_Z$ scale they are related as:

$$\frac{1}{\alpha_3(M_Z)} = \frac{1}{\alpha_2(M_Z)} + \frac{b_3 - b_2}{b_1 - b_2}\left[\frac{1}{\alpha_1(M_Z)} - \frac{1}{\alpha_2(M_Z)}\right].$$

Where $b_i$ are the $\beta$ function coefficients for the couplings $\alpha_i$, respectively.

(b) Keeping in mind that the masses of the quarks are known to be $m_{u,d} \sim \mathcal{O}(1)$MeV with the other ones much heavier ($m_s, m_c, m_b, m_t \sim 10^2, 10^3, 4 \times 10^3, 1.7 \times 10^5$ MeV respectively), write down the low-energy QCD Lagrangian including only the $u$ and $d$ quarks consistent with the gauge symmetry $SU(3)_c \times U(1)_{EM}$. Show that baryon number is an accidental symmetry of this Lagrangian.

(c) Identify the approximate (chiral) symmetries of this Lagrangian in the limit of $m_{u,d} \to 0$ and argue why they can be broken.

(i) Construct an effective field theory for scalar fields that captures the desired symmetry breaking. Identify the unbroken symmetry group and the identity of the pseudo-Goldstone modes of this approximate symmetry.

(ii) Write down an effective Lagrangian for the pseudo-Goldstone bosons for energies $E \ll \Lambda_{QCD}$.

(iii) Could this approximate symmetry explain the mass difference between protons and neutrons?

(d) Is it possible to extend this analysis to include also the $s$ and $c$ quarks? Explain.

**END OF PAPER**