MAT3

MATHEMATICAL TRIPOS Part III

Tuesday, 6 June, 2023 1:30 pm to 4:30 pm

PAPER 304

ADVANCED QUANTUM FIELD THEORY

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

Attempt no more than **THREE** questions.
There are **FOUR** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

The action for the harmonic oscillator of unit mass in quantum mechanics is

$$S[x] = \int dt \left(\frac{1}{2}\dot{x}^2(t) - \frac{1}{2}\omega^2 x^2(t) \right).$$

(a) Show, by Fourier transform or otherwise, that the propagator of the theory is

$$D(t-t') = \frac{1}{2\omega} e^{i\omega|t-t'|}.$$

By introducing a source $J(t)$, show that the generating functional $Z_0[J]$ of the harmonic oscillator may be written as

$$Z_0[J] = Z_0[0] \exp\left(-\frac{1}{2} \int dt \int dt'\, J(t) D(t-t') J(t') \right).$$

(b) An anharmonic term

$$S_\lambda[x] = -\frac{\lambda}{3!} \int dt\, x^3(t),$$

is now added to the action. Write down an expression for the generating functional $Z[J]$ of the interacting theory in terms of an infinite series involving the generating functional of the harmonic oscillator $Z_0[J]$ and its (functional) derivatives. Briefly comment on the convergence of this series.

(c) Write down the Feynman rules for the anharmonic oscillator. Hence, find an integral expression for the *connected* correlation function $\langle T\{x(t_1)x(t_2)\}\rangle_{\text{conn}}$ up to order λ^2. Without evaluating the integral, do you expect this correlation function to diverge? Briefly explain your reasoning.

2

This question is about the massless real scalar theory in four-dimensional Minkowski space with Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi \, \partial^\mu \phi - \frac{\lambda}{4!}\phi^4 .$$

(a) Without derivation, write down the momentum space Feynman rules for the theory (including counterterms). Draw the diagrams that contribute to the *renormalized* 1PI four-point function Γ_4, up to order λ^2.

(b) Γ_4 may be written in terms of the Mandelstam variables s, t and u. We define the effective coupling λ_{eff} at an energy scale M as

$$-i\lambda_{\text{eff}} = \Gamma_4, \qquad \text{when} \qquad s = t = u = -M^2 .$$

By evaluating the relevant integrals using dimensional regularization, show that

$$-i\lambda_{\text{eff}} = -i\lambda - \frac{i(-i\lambda)^2}{32\pi^2}\int_0^1 dx \left(\frac{6}{\epsilon} - 3\gamma + ...\right) - i\delta\lambda + ...,$$

Hence find a suitable choice of the counterterm $\delta\lambda$ such that $\lambda_{\text{eff}} = \lambda$ at renormalization scale M.

(c) State, without derivation, the Callan-Symanzik equation for this theory and, assuming there is no wavefunction renormalization at this order in λ, show that the beta function for λ is given by

$$\beta(\lambda) = \frac{3\lambda^2}{16\pi^2} + ...$$

where $+...$ denotes terms of order λ^3 and higher.

[Hints: You may use the following results for the d-dimensional integrals with integers a and b:

$$\int_{\mathbb{R}^d} \frac{d^d\ell}{(2\pi)^d}\frac{(\ell^2)^a}{(\ell^2+\Delta)^b} = \frac{\Gamma(b-a-d/2)\Gamma(a+d/2)}{(4\pi)^{d/2}\Gamma(b)\Gamma(d/2)}\Delta^{-(b-a-d/2)},$$

as well as the Laurent expansion of the Γ-function near its pole at $z = 0$:

$$\Gamma(z) = \frac{1}{z} - \gamma + \mathcal{O}(z),$$

where γ is the Euler-Mascheroni constant.]

3

A Yang-Mills vector field A_μ has field strength

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu].$$

(a) State how $F_{\mu\nu}$ transforms under gauge transformations and show that the Lagrangian

$$\mathcal{L} = -\frac{1}{4}\mathrm{Tr}(F_{\mu\nu}F^{\mu\nu}),$$

is gauge-invariant. Briefly explain why gauge-fixing is necessary to find the propagator.

(b) A scalar field $\phi(x)$ in the adjoint representation transforms under the gauge symmetry as

$$\phi \to U^\dagger \phi U.$$

Show that the covariant derivative of ϕ is given by

$$D_\mu\phi = \partial_\mu\phi - ig[A_\mu, \phi].$$

By writing $A_\mu = A^a_\mu(x)T_a$ and $\phi(x) = \phi^a(x)T_a$ and

$$U(x) = \exp\left(i\alpha^a(x)T_a\right),$$

where T_a are generators of the gauge group, find the infnitesimal form (i.e. to first order in $\alpha^a(x)$) of the gauge transformations of $\phi^a(x)$ in terms of the fields and structure constants of the gauge group.

(c) Suppose we also have a fermion $\psi(x)$, which transforms in the fundamental representation of the gauge group.

(i) Write down the general gauge-invariant Lagrangian in d dimensions, up to and including terms quartic in the fields $\psi(x)$, $\phi(x)$ and $A_\mu(x)$, that is invariant under the transformations $\phi \to -\phi$, $\psi \to -\psi$. Clearly state the mass dimension of any coupling constants you introduce and comment on how each term is expected to evolve under renormalization group flow .

(ii) Draw the Feynman diagrams that contribute to the four-scalar 1PI correlation function to one loop (you do not have to evaluate the diagrams or include counterterms).

4

A non-abelian gauge theory may be described by the Lagrangian

$$\mathcal{L} = -\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} + \bar{\psi}(i\not{D} - m)\psi + \bar{c}_a\left(\frac{\delta G^a}{\delta A^b_\mu}\right)D^{bc}_\mu c_c + B_a G^a - \frac{\xi}{2}B^a B_a$$

where $G^a(A_\mu)$ is a fixing function, such as $\partial^\mu A^a_\mu$, the covariant derivative is $D_\mu = \partial_\mu - igA_\mu$ and all other terms have their usual meanings.

(a) By considering the variation of each field, show that the following BRST transformation is such that $Q^2(\Phi) := Q(Q(\Phi)) = 0$ for all fields Φ (i.e. it is nilpotent)

$$\begin{aligned} Q(A^a_\mu) &= D^{ab}_\mu c_b, \qquad Q(\psi) = igc^a T_a \psi \\ Q(c^a) &= -\frac{1}{2}g f^a_{bc} c^b c^c, \qquad Q(\bar{c}^a) = B^a, \qquad Q(B^a) = 0. \end{aligned}$$

You may use the Jacobi identity $f_{d[a}{}^e f_{bc]}{}^d = 0$ without proof.

(b) By considering the fermionic function,

$$\Psi := \bar{c}_a\left(G^a(A) - \frac{\xi}{2}B^a\right),$$

show that the Lagrangian $\mathcal{L}$ is invariant under the above BRST transformation.

(c) Assume that the BRST transformation can be realised by some operator $\widehat{\mathcal{Q}}$ so that

$$Q(\Phi) = [\widehat{\mathcal{Q}}, \Phi]_\pm := \widehat{\mathcal{Q}}\Phi \pm \Phi\widehat{\mathcal{Q}}$$

where the plus (minus) applies if Φ is fermionic (bosonic). A change of gauge-fixing corresponds to a change in the gauge-fixing function Ψ. By requiring the transition amplitude between physical states $|i\rangle$ and $|f\rangle$,

$$\langle f|i\rangle = \int_i^f \mathcal{D}\Phi\, e^{iS_0[\Phi] + i\int d^4x\,\{\widehat{\mathcal{Q}},\Psi\}}$$

to be invariant under a change of gauge choice, explain why physical states must satisfy $\widehat{\mathcal{Q}}|i\rangle = 0 = \widehat{\mathcal{Q}}|f\rangle$.

END OF PAPER