UNIVERSITY OF CAMBRIDGE

MAMA/303, NST3AS/303, MAAS/303

MAT3

MATHEMATICAL TRIPOS Part III

Wednesday, 7 June, 2023 9:00 am to 11:00 am

PAPER 303

STATISTICAL FIELD THEORY

Before you begin please read these instructions carefully

Candidates have TWO HOURS to complete the written examination.

Attempt no more than **TWO** questions.
There are **THREE** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(a) Explain briefly how the Landau-Ginzburg approach extends Landau's theory of phase transitions.

(b) Consider a theory involving a real scalar field $\phi(\boldsymbol{x})$ in d dimensions with a free energy functional of the form

$$F[\phi] = \frac{1}{2}\int d^d x \Big[\gamma\, \boldsymbol{\nabla}\phi \cdot \boldsymbol{\nabla}\phi + \mu^2\, \phi^2\Big]\,,$$

where $\mu^2 \sim (T - T_c)$ and $\gamma > 0$.

(i) Show that

$$F[\phi_{\boldsymbol{k}}] = \frac{1}{2}\int \frac{d^d k}{(2\pi)^d}\Big[\gamma k^2 + \mu^2\Big]\,|\phi_{\boldsymbol{k}}|^2\,,$$

where

$$\phi(\boldsymbol{x}) = \int \frac{d^d k}{(2\pi)^d}\, e^{i\boldsymbol{k}\cdot\boldsymbol{x}}\, \phi_{\boldsymbol{k}}\,.$$

(ii) Show that this free energy leads to a heat capacity per unit volume,

$$c = \frac{1}{2}\int \frac{d^d k}{(2\pi)^d}\left[1 - \frac{g(T)}{\gamma k^2 + \mu^2} + \frac{h(T)}{(\gamma k^2 + \mu^2)^2}\right]\,,$$

where $g(T)$ and $h(T)$ are functions of T that you should determine.

(iii) Hence argue that $c \sim |T - T_c|^{-\alpha}$ for $T \to T_c^+$ when $d < 4$, where α is a critical exponent that you should determine.

(c) Now instead consider a theory which has an effective free energy,

$$f(T, m) = a_2(T)\, m^2 + a_3\, m^3 + a_4\, m^4\,,$$

in the mean field approximation, where m is the magnetisation, $a_2(T) \sim (T - T_c)$, $a_3 \neq 0$ and $a_4 > 0$. Find relations between the coefficients a_2, a_3 and a_4 which determine whether the state is ordered or disordered.

Can this system possess a second-order (continuous) phase transition? Can it possess a first-order phase transition? You should justify your answers.

2

Consider a theory involving a real scalar field ϕ in d dimensions with a free energy of the form

$$F[\phi] = \int d^d x \left[\tfrac{1}{2} \nabla\phi \cdot \nabla\phi + \tfrac{1}{2}\mu_0^2\, \phi^2 + \ldots \right] . \qquad (*)$$

(a) Describe the three steps of the renormalisation group procedure (in momentum space) for such a free energy and explain how they result in a flow of the parameters in the free energy. You should denote the original cutoff Λ and the new cutoff Λ/ζ.

(b) Now suppose that the free energy contains quadratic terms in ϕ, as in equation $(*)$, along with

$$\sim \int d^d x \, g_{n,m}\, \phi^n \left(\nabla^2 \phi\right)^m ,$$

where n and m are positive integers. What is the naive (engineering) dimension of the coupling $g_{n,m}$? Why might the scaling dimension differ from this? What must the dimension d be (in terms of n and m) if the coupling is marginal?

(c) Now instead suppose that the free energy contains quadratic terms in ϕ, as in equation $(*)$, along with

$$\sim \int d^d x \left[g_0\, \phi^4 + \lambda_0\, \phi^6 \right] .$$

(i) Draw Feynman diagrams representing the corrections to $\mu^2(\zeta)$ from the $g_0\, \phi^4$ and $\lambda_0\, \phi^6$ terms up to and including order g_0^2, $\lambda_0 g_0$, λ_0.

(ii) Calculate the contributions $\sim \lambda_0$ and $\sim \lambda_0 g_0$ to the flow of the coupling, $\mu^2(\zeta)$. [You may assume that $\langle \phi^+_{\boldsymbol{k}}\, \phi^+_{\boldsymbol{k}'} \rangle_+ = (2\pi)^d\, \delta^{(d)}(\boldsymbol{k} + \boldsymbol{k}') G_0(k)$, where $G_0(k) = 1/(k^2 + \mu_0^2)$, for appropriately defined ϕ^+ and $\langle \ldots \rangle_+$, you may use Wick's theorem without proof, and you may leave your final answer in integral form. You may ignore other corrections, including those from the rescaling of the field.]

3

Consider an $O(N)$ model involving an N-component real field $\boldsymbol{\phi}(\boldsymbol{x})$ in d dimensions with free energy,

$$F_1[\boldsymbol{\phi}] = \int d^d x \Big[\tfrac{1}{2}\gamma\,(\partial_i \phi_a)(\partial_i \phi_a) + \tfrac{1}{2}\mu^2\, \boldsymbol{\phi}\cdot\boldsymbol{\phi} + g\,(\boldsymbol{\phi}\cdot\boldsymbol{\phi})^2 \Big]\,,$$

where $\gamma > 0$, $g > 0$, $\mu^2 \sim (T - T_c)$, $\boldsymbol{\phi}\cdot\boldsymbol{\phi} = \phi_a \phi_a$, repeated indices are summed over, $i = 1, 2, \ldots, d$ and $a = 1, 2, \ldots, N$.

(a) Determine the mean field theory ground state of $\boldsymbol{\phi}$, $\boldsymbol{\phi}_0$, for $T > T_c$ and $T < T_c$. Determine the number of gapless modes in each case and interpret this in the context of any relevant symmetries.

(b) Determine the correlation function for the gapless mode, $\theta(\boldsymbol{x})$, in the $O(2)$ model, $\langle \theta(\boldsymbol{x})\theta(\boldsymbol{y})\rangle$, for $T < T_c$. [You may assume that for a real scalar field $\tilde{\phi}(\boldsymbol{x})$ with free energy,

$$\tilde{F}[\tilde{\phi}] = \int d^d x \Big[\tfrac{1}{2}\tilde{\gamma}\, \boldsymbol{\nabla}\tilde{\phi}\cdot\boldsymbol{\nabla}\tilde{\phi} + \tfrac{1}{2}\tilde{\mu}^2\, \tilde{\phi}^2 \Big]\,,$$

the correlation function is, $\big\langle \tilde{\phi}(\boldsymbol{x})\tilde{\phi}(\boldsymbol{y})\big\rangle = \int \frac{d^d k}{(2\pi)^d} \frac{e^{-i\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{y})}}{\tilde{\gamma}k^2 + \tilde{\mu}^2}$.] What is the correlation length? Determine the asymptotic behaviour ($r \to \infty$) of $\langle \theta(\boldsymbol{x})\theta(\boldsymbol{y})\rangle$ as a function of $r = |\boldsymbol{x} - \boldsymbol{y}|$.

Using this determine whether an ordered phase will exist in $d = 1, 2, 3, 4$ and justify your answer. What is the lower critical dimension?

(c) Now instead consider a 3-component real scalar field $\boldsymbol{\phi}(\boldsymbol{x})$ in d dimensions with free energy,

$$F_2[\boldsymbol{\phi}] = \int d^d x \Big[\tfrac{1}{2}\gamma\,(\partial_i \phi_a)(\partial_i \phi_a) + \tfrac{1}{2}\mu^2\, \phi_1^2 + \tfrac{1}{2}\lambda^2\, \phi_2^2 + \tfrac{1}{2}\lambda^2\, \phi_3^2 + g\,(\boldsymbol{\phi}\cdot\boldsymbol{\phi})^2 \Big]\,,$$

where $\gamma > 0$, $g > 0$, $\mu^2 \neq 0$, $\lambda^2 \neq 0$, $\boldsymbol{\phi}\cdot\boldsymbol{\phi} = \phi_a\phi_a$, repeated indices are summed over, $i = 1, 2, \ldots, d$ and $a = 1, 2, 3$. What are the symmetries of the free energy?

Determine the mean field theory ground state of $\boldsymbol{\phi}$ and describe the pattern of spontaneous symmetry breaking for each of the following cases:

(i) $\mu^2, \lambda^2 > 0$,

(ii) $\mu^2 < 0$, $\lambda^2 > 0$,

(iii) $\mu^2 > 0$, $\lambda^2 < 0$,

(iv) $\mu^2 = \lambda^2 < 0$.

END OF PAPER