MAMA/302, NST3AS/302, MAAS/302

MAT3

MATHEMATICAL TRIPOS Part III

Monday, 5 June, 2023 9:00 am to 12:00 pm

PAPER 302

SYMMETRIES, FIELDS AND PARTICLES

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

Attempt no more than **THREE** questions.
There are **FOUR** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Consider the orthogonal group $O(3) = \left\{ M \in GL(3,\mathbb{R}) \;\middle|\; M^{\mathsf{T}} M = I \right\}$.

(a) Show that the set of matrices $\{ g(\theta) \mid 0 \leqslant \theta < 2\pi \}$, such that

$$g(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

forms a 1-parameter subgroup of $O(3)$.

(b) Let $L(O(3))$ be the Lie algebra of $O(3)$. What conditions must the matrices $X \in L(O(3))$ satisfy?

(c) The adjoint representation of $O(3)$ is defined to be the map given by

$$\mathrm{Ad}_A(X) = AXA^{\mathsf{T}}$$

for all $A \in O(3)$ and all $X \in L(O(3))$. Show that Ad_A is indeed a map $L(O(3)) \to L(O(3))$ and that the map preserves the group product, i.e. it is a group homomorphism.

(d) Given a group representation D, how can one define a corresponding algebra representation d? Verify that your definition satisfies the properties of a Lie algebra representation.

(e) Construct $\mathrm{ad} : L(O(3)) \to \mathfrak{gl}(L(O(3)))$, the adjoint representation of $L(O(3))$.

(f) Discuss how the exponential map relates elements of $L(O(3))$ to elements of $O(3)$. Given a representation d of $L(O(3))$, can you construct a corresponding representation of $O(3)$?

2

This question concerns finite-dimensional, irreducible representations d of the $SU(2)$ Lie algebra. Let the *Hermitian* generators of the $SU(2)$ Lie algebra be denoted J_1, J_2, J_3, with $[J_i, J_j] = i\epsilon_{ijk}J_k$. Let I be the maximum weight of the irreducible representation d, and denote any eigenvector of $d(J_3)$ by v_m, such that

$$d(J_3)\, v_m = m\, v_m \,.$$

(a) Construct the J_3-eigenvector basis for the vector space on which d acts, showing that the dimension of this space is equal to $2I + 1$. [Hint: use $J_\pm = J_1 \pm iJ_2$.]

(b) Writing the vector norm $\|v_m\| = N_m$ in terms of the inner product $v_m^\dagger v_m = N_m^2$, define normalized eigenvectors

$$|I, m\rangle := \frac{1}{N_m}\, v_m \,.$$

Show that the normalized states may be expressed in the form

$$|I, m\rangle = A(I, m)\, (d(J_-))^{I-m}\, |I, I\rangle \;,$$

where $A(I, m)$ is a numerical factor depending on I and m which you should determine. [Hint: Note that $J_-^\dagger = J_+$.]

(c) An operator C maps the $SU(2)$ generators as follows

$$\mathsf{C}J_1\mathsf{C}^{-1} = -J_1\,, \quad \mathsf{C}J_2\mathsf{C}^{-1} = +J_2\,, \quad \mathsf{C}J_3\mathsf{C}^{-1} = -J_3\,.$$

Show that the set $\{\, J_i' \,\}$, where $J_i' := \mathsf{C}J_i\mathsf{C}^{-1}$, also generates an $SU(2)$ algebra.

(d) Because the up and down quarks have very light masses, the strong force has an approximate $SU(2)$ symmetry called isospin symmetry. In this question we assume that this is an exact symmetry. The three π mesons form the three components of an $SU(2)$ triplet ($I = 1$) $\left\{\, |\pi^-\rangle\,, \left|\pi^0\right\rangle, |\pi^+\rangle \,\right\}$, corresponding to $m = -1$, 0, and 1, respectively. Given that $\mathsf{C}\left|\pi^0\right\rangle = \left|\pi^0\right\rangle$, how are $\mathsf{C}\,|\pi^+\rangle$ and $\mathsf{C}\,|\pi^-\rangle$ related to the original states?

3

This question concerns a simple, complex Lie algebra denoted by $\mathfrak{g}$ (or by $L(G)$).

(a) Briefly explain the notions of **Cartan subalgebra**, **roots**, the **Cartan–Weyl basis**, **simple roots**, and the **Cartan matrix**, all with respect to $\mathfrak{g}$.

(b) Prove that any positive root can be written *uniquely* as a linear combination of simple roots, with positive integer coefficients.

For the rest of this question, we consider the Lie algebra with Cartan matrix

$$A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} . \qquad (*)$$

(c) Using the Cartan matrix $(*)$ and denoting the simple roots by $(\alpha_{(1)}, \alpha_{(2)}, \alpha_{(3)})$, determine the full root set Φ and its size $|\Phi|$.

(d) Working in the basis of the fundamental weights $\omega_{(1)}$, $\omega_{(2)}$, $\omega_{(3)}$, where a weight vector corresponding to $q_1\omega_{(1)} + q_2\omega_{(2)} + q_3\omega_{(3)}$ is denoted by $|q_1, q_2, q_3\rangle$ (i.e. by its Dynkin labels), find all the weights of the representations whose highest weights are, respectively, (i) $|1, 0, 0\rangle$ and (ii) $|0, 1, 0\rangle$.

[Hints: you do not need to know A^{-1}, nor do you need to use a specific basis for the root vectors. You may assume that there are no degeneracies.]

(e) Referring to the Cartan matrix $(*)$, consider the following three sets of vectors labelled by A, B, and C.

$$\begin{aligned} \alpha^A_{(1)} &= (1, -1, 0) \\ \alpha^A_{(2)} &= (-1, 0, 1) \\ \alpha^A_{(3)} &= (0, 1, -1) \end{aligned} \quad , \qquad \begin{aligned} \alpha^B_{(1)} &= (1, 1, 0) \\ \alpha^B_{(2)} &= (-1, 0, 1) \\ \alpha^B_{(3)} &= (1, -1, 0) \end{aligned} \quad , \qquad \begin{aligned} \alpha^C_{(1)} &= (1, -1, 0, 0) \\ \alpha^C_{(2)} &= (0, 1, -1, 0) \\ \alpha^C_{(3)} &= (0, 0, 1, -1) \end{aligned}$$

Determine whether each of the sets $\{\, \alpha^A_{(i)} \,\}$, $\{\, \alpha^B_{(i)} \,\}$, and $\{\, \alpha^C_{(i)} \,\}$ is a valid or invalid set of simple roots.

4

Let G be a compact, matrix Lie group with matrix Lie algebra $\mathfrak{g}$ (or $L(G)$). Consider a field $\phi(x)$ (in Minkowski spacetime) which transforms in the fundamental representation of G under gauge transformations, that is,

$$\phi(x) \mapsto \phi'(x) = g(x)\phi(x)\,.$$

for $g(x) \in G$.

(a) Using the fundamental covariant derivative $D_\mu\phi := (\partial_\mu + A_\mu)\phi$, and requiring that $D_\mu\phi \mapsto gD_\mu\phi$, show that the gauge field $A_\mu(x)$ must transform as

$$A_\mu \mapsto A'_\mu = gA_\mu g^{-1} - (\partial_\mu g)g^{-1}\,.$$

(b) Given that $A_\mu(x) \in \mathfrak{g}$, show that $A'_\mu(x) \in \mathfrak{g}$.

(c) The field strength tensor is given by

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]\,.$$

By showing that $F_{\mu\nu}\phi = [D_\mu, D_\nu]\phi$, or otherwise, find how $F_{\mu\nu}$ transforms under gauge transformations.

(d) Suppose that $\mathfrak{g}$ is semisimple, with Killing form κ. Consider the following Lagrangian densities

$$\mathcal{L}_1 = (\kappa(F_{\mu\nu}, F^{\mu\nu}))^q\,, \quad q \in \mathbb{Z}^+$$
$$\mathcal{L}_2 = \kappa(\mathrm{D}_\mu F_{\nu\rho}, \mathrm{D}^\mu F^{\nu\rho})$$

where D_μ is the adjoint covariant derivative

$$\mathrm{D}_\mu F_{\nu\rho} := \partial_\mu F_{\nu\rho} + [A_\mu, F_{\nu\rho}]\,.$$

Show that both $\mathcal{L}_1$ and $\mathcal{L}_2$ are gauge-invariant.

[Hint: You may use without proof that $\kappa([X,Y],Z) = \kappa(X,[Y,Z])$ for X, Y, $Z \in \mathfrak{g}$.]

END OF PAPER