MAMA/301, NST3AS/301, MAAS/301, NST3PHY/QFT, MAPY/QFT

MAT3

MATHEMATICAL TRIPOS Part III

Thursday, 1 June, 2023 9:00 am to 12:00 pm

PAPER 301

QUANTUM FIELD THEORY

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

Attempt no more than **THREE** questions.
There are **FOUR** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Consider a relativistic theory of a single scalar field $\phi(x)$ with Lagrangian density $\mathcal{L}$ and action,

$$S = \int d^4x \, \mathcal{L}$$

Define the following: a *continuous symmetry* of S, a *conserved current* and the corresponding *conserved charge*. Explain carefully how these three concepts are related. Your answer should discuss the circumstances in which the existence of one of these three implies the other two.

Define the energy-momentum tensor of the theory, $T^{\rho\sigma}$, and find a formula for it which holds in the case,

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi)$$

for any potential V. Explain why the energy momentum tensor corresponds to four conserved currents and find the corresponding conserved charges.

By considering the invariance of S under infinitesimal Lorentz transformations, show that,

$$\partial_\mu \left[x^\rho T^{\mu\sigma} - x^\sigma T^{\mu\rho} \right] = 0$$

Hence show that the three-vector quantity,

$$V^i := \frac{d}{dt} \int d^3x \, x^i T^{00}$$

is constant in time.

2 Consider a theory of two complex scalar fields, Ψ_1 and Ψ_2 with masses M_1 and M_2 respectively, interacting with a real scalar field ϕ of mass μ. The Lagrangian density has the form $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{\rm int}$ where $\mathcal{L}_0$ is the sum of free Lagrangians for each field and the interaction term is,

$$\mathcal{L}_{\rm int} \quad = \quad -g \sum_{i=1,2} \phi\, \Psi_i^* \Psi_i$$

where g is a coupling constant.

Using Dyson's formula and Wick's theorem, calculate the tree-level scattering amplitude for the pair-creation process,

$$\Psi_1(p) + \bar{\Psi}_1(q) \quad \rightarrow \quad \Psi_2(p') + \bar{\Psi}_2(q')$$

where $\Psi_i(p)$ $(\bar{\Psi}_i(q))$ denotes a particle (anti-particle) with on-shell four-momentum $p^\mu = (p^0, \mathbf{p})$ $(q^\mu = (q^0, \mathbf{q}))$ of the Ψ_i field for $i = 1, 2$.

Working in the COM frame where $\mathbf{p} + \mathbf{q} = 0$, find the minimum value of $|\mathbf{p}|$ for which the scattering amplitude is non-zero.

3 Define the Dirac spinor representation of the Lorentz group. In particular, you should give a formula for the representative $\mathcal{S}[\Lambda]$ of an element Λ of the Lorentz group of the form,

$$\Lambda \;=\; \mathrm{Exp}\left[\frac{1}{2}\Omega_{\rho\sigma}\mathcal{M}^{\rho\sigma}\right]$$

where Exp denotes the exponential function for matrices, $\Omega_{\rho\sigma} = -\Omega_{\sigma\rho}$ is a matrix of parameters and

$$(\mathcal{M}^{\rho\sigma})^{\mu\nu} \;=\; \eta^{\rho\mu}\eta^{\sigma\nu} - \eta^{\sigma\mu}\eta^{\rho\nu}$$

are the generators of infinitesimal Lorentz transformations.

Using a suitable representation of the Dirac γ-matrices, find $S[\Lambda]$ explicitly in the cases,

$$i)\;\; \Lambda = R(\phi) := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\phi & \sin\phi & 0 \\ 0 & -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad ii)\;\; \Lambda = B(\beta) := \begin{pmatrix} \cosh\beta & \sinh\beta & 0 & 0 \\ \sinh\beta & \cosh\beta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Determine the value of $\mathcal{S}[R(\phi)]$ at $\phi = 2\pi$ and comment on the significance of your result.

The standard basis of positive frequency solutions of the Dirac equation is given in an inertial frame with coordinates $x^\mu = (t, x, y, z)$ as, $u(p)\exp(-ip\cdot x)$, where $u(p)$ is the constant four-component complex vector,

$$u(p) \;=\; \begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi \\ \sqrt{p\cdot\bar{\sigma}}\,\xi \end{pmatrix}$$

with $\sigma^\mu = (\mathbb{I}_2, \vec{\sigma})$ and $\bar{\sigma}^\mu = (\mathbb{I}_2, -\vec{\sigma})$ and $\xi \in \mathbb{C}^2$ an arbitrary constant vector. Here $\mathbb{I}_2$ is the (2×2) unit matrix and $\vec{\sigma} = (\sigma^1, \sigma^2, \sigma^3)$ where σ^i, with $i = 1, 2, 3$ are the Pauli matrices. Consider the constant spinor $u(p)$ in two different inertial frames: the particle rest frame where $p^\mu = p_1^\mu := (m, 0, 0, 0)$ and another frame where $p^\mu = p_2^\mu := (m\cosh\beta, m\sinh\beta, 0, 0)$. Show explicitly that,

$$u(p_2) \;=\; \mathcal{S}[B(\beta)]\cdot u(p_1)$$

4 What is meant by a *gauge symmetry*? Write down the Lagrangian density for Quantum Electrodynamics (QED) and demonstrate its gauge invariance explicitly.

Write down (without derivation) a set of Feynman rules for calculating scattering amplitudes in QED in a covariant gauge (e.g. Feynman gauge).

Draw all tree-level diagrams contributing to the following two-particle scattering processes involving electrons (e^-), positrons (e^+) and photons (γ):

i) $e^+ + e^- \to e^+ + e^-$,

ii) $e^+ + e^- \to \gamma + \gamma$,

iii) $\gamma + e^+ \to \gamma + e^+$.

In each case you should identify clearly the quantum numbers characterising the initial and final state and label each diagram accordingly.

For each of these processes, use your Feynman rules to evaluate the corresponding tree-level scattering amplitude.

END OF PAPER