UNIVERSITY OF CAMBRIDGE

MAMA/211, NST3AS/211, MAAS/211

MAT3

MATHEMATICAL TRIPOS Part III

Friday, 9 June, 2023 1:30 pm to 4:30 pm

PAPER 211

ADVANCED FINANCIAL MODELS

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

Attempt **ALL** questions.
There are **FOUR** questions in total.
Question 2 is worth twice as much as each of the other three questions.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Consider a one period model of n assets with prices $(P_t)_{t\in\{0,1\}}$ and no dividends.

(a) What does it mean to say a vector $H \in \mathbb{R}^n$ is an *arbitrage*? a *terminal consumption arbitrage*?

(b) What does it mean to say a vector $\eta \in \mathbb{R}^n$ is a *numéraire portfolio*? Prove that if there exist both an arbitrage and a numéraire portfolio, then there exists a terminal consumption arbitrage.

Henceforth, suppose that P_1 has the normal distribution with mean vector μ and covariance matrix V. Let

$$\mathcal{A} = \{Vx : x \in \mathbb{R}^n\} \subseteq \mathbb{R}^n$$

be the image of V.

(c) Show that if $\mathcal{A} = \mathbb{R}^n$ then there is no arbitrage.

(d) Show that if there exists a numéraire portfolio, then a risk-free bond can be replicated.

(e) Suppose there exists a numéraire portfolio η and let

$$r = \frac{\eta \cdot \mu}{\eta \cdot P_0} - 1.$$

Show that there is no arbitrage if and only if $\mu - (1+r)P_0 \in \mathcal{A}$.

2 Consider a one-period model of n assets with prices $(P_t)_{t\in\{0,1\}}$ and no dividends.

(a) What does it mean to say $Y = (Y_t)_{t\in\{0,1\}}$ is a martingale deflator for the model? Show that if Y^0, Y^1 are martingale deflators and $\varepsilon^0, \varepsilon^1$ are positive real constants, then $\varepsilon^0 Y^0 + \varepsilon^1 Y^1$ is also a martingale deflator.

Now assume that there exists at least one martingale deflator.

(b) Let ξ_0 be a real number and ξ_1 a random variable such that $\mathbb{E}(\xi_1 Y_1) < \xi_0 Y_0$ for every martingale deflator Y. Use the one-period fundamental theorem of asset pricing to show that there exists a portfolio $H \in \mathbb{R}^n$ such that

$$H \cdot P_0 \leqslant \xi_0 \text{ and } H \cdot P_1 \geqslant \xi_1 \text{ almost surely.}$$

(c) Let ξ_0 be a real number and ξ_1 a random variable such that $\mathbb{E}(\xi_1 Y_1) \leqslant \xi_0 Y_0$ for every martingale deflator Y, with equality for at least one such Y. Show that there exists a portfolio $H \in \mathbb{R}^n$ such that

$$H \cdot P_0 = \xi_0 \text{ and } H \cdot P_1 = \xi_1 \text{ almost surely.}$$

Now let $X_0 > 0$ be a fixed initial wealth and let $u : (0, \infty) \to \mathbb{R}$ be smooth, strictly increasing, strictly concave utility function with

$$\lim_{x\downarrow 0} u'(x) = \infty \text{ and } \lim_{x\uparrow\infty} u'(x) = 0.$$

For each $y > 0$, let

$$\hat{u}(y) = \max_{x>0}\{u(x) - xy\}.$$

(d) Show that

$$\mathbb{E}[u(H \cdot P_1)] \leqslant \mathbb{E}[\hat{u}(Y_1)] + X_0 Y_0$$

for all martingale deflators Y and for all $H \in \mathbb{R}^n$ such that $H \cdot P_0 = X_0$. Show that there is equality if $u'(H \cdot P_1) = Y_1$ almost surely.

(e) Let Y^* minimise the quantity $\mathbb{E}[\hat{u}(Y_1)] + X_0 Y_0$ among all martingale deflators Y. By considering the expression $\mathbb{E}[\hat{u}(Y_1^* + \varepsilon Y_1)] + X_0(Y_0^* + \varepsilon Y_0)$ or otherwise, show that

$$\mathbb{E}[\hat{u}'(Y_1^*)Y_1] + X_0 Y_0 \geqslant 0$$

for all martingale deflators Y, with equality if $Y = Y^*$. [You may pass derivatives inside expectations without justification.] Conclude that there exists an $H^* \in \mathbb{R}^n$ such that $H^* \cdot P_0 = X_0$ and

$$\mathbb{E}[u(H^* \cdot P_1)] = \mathbb{E}[\hat{u}(Y_1^*)] + X_0 Y_0^*.$$

[Hint: check that $y = u'(x)$ if and only if $x = -\hat{u}'(y)$.]

3

Consider a discrete-time market with a family of risk-free bonds of maturity T for $T \in \mathcal{T}$ and time-t price P_t^T, an asset with time-t price S_t, and a family of European call options of maturity $T \in \mathcal{T}$ and strikes $K \in \mathcal{K}$ written on the asset with time-t price $C_t^{T,K}$, where $\mathcal{T} = \{1, 2, \ldots, T_M\}$ and $\mathcal{K} = \{K_1, \ldots, K_N\}$ are given finite sets. Assume that there is no arbitrage. Throughout this question you may *not* use the fundamental theorem of asset pricing.

(a) By considering the stopping time

$$\tau = \inf\{t \geqslant 0 : P_t^T \leqslant 0\}$$

show that $P_t^T > 0$ almost surely for all $0 \leqslant t < T$.

(b) Show that $K \mapsto C_t^{T,K}$ is non-increasing almost surely for all $0 \leqslant t < T$.

(c) Consider the case where the asset pays no dividend and $P_{t-1}^t \leqslant 1$ almost surely for all $t \in \mathcal{T}$. Show in this case that $T \mapsto C_t^{T,K}$ is non-decreasing for all $K \in \mathcal{K}$ and $0 \leqslant t < T_M$.

(d) Fix $T \in \mathcal{T}$ and consider a European contingent claim with time-T payout $\xi_T = g(S_T)$. Show, in the case that $S_T \in \mathcal{K}$ almost surely, that the time-t no-arbitrage price of the claim is

$$\pi_t = g(K_1) P_t^T + \sum_{i=2}^{N} \frac{g(K_i) - g(K_{i-1})}{K_i - K_{i-1}} (C_t^{T,K_{i-1}} - C_t^{T,K_i}).$$

4 Consider a market with a bank account and one stock. The risk-free interest rate is a constant r and the stock price process S is positive and evolves as

$$dS_t = S_t \ \mu \ dt + S_t \ \sigma \ dW_t$$

where W is a Brownian motion and μ and σ are real constants with $\sigma > 0$. Given $x \in \mathbb{R}$ and a (sufficiently integrable) previsible process θ let $X^{x,\theta}$ be the time-t wealth of a self-financing investor who has initial wealth x and holds θ_t shares of the stock at time t. Fix a non-random time horizon $T > 0$.

(a) Show that

$$dX_t^{x,\theta} = r(X_t - \theta_t S_t)dt + \theta_t dS_t.$$

(b) Show that there is a risk-neutral measure $\mathbb{Q}$ with density

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = e^{-\lambda^2 T/2 - \lambda W_T}$$

for a constant λ to be determined in terms of the constants μ, r, σ.

(c) Fix a bounded smooth function g, and let

$$x = \int_{-\infty}^{\infty} e^{-rT} g(S_0 e^{(r-\sigma^2/2)T + \sigma\sqrt{T}z})\varphi(z)dz$$

and

$$\theta_t = \int_{-\infty}^{\infty} g'(S_t e^{(r+\sigma^2/2)(T-t)+\sigma\sqrt{T-t}z})\varphi(z)dz$$

for $0 \leqslant t \leqslant T$, where $\varphi(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$ is the standard normal density. Show that

$$X_T^{x,\theta} = g(S_T).$$

[You may use standard results from stochastic calculus without justification. You may assume that any PDE that you encounter of the form

$$\frac{\partial V}{\partial t} + \mathcal{L}V = 0,$$

where $\mathcal{L}$ is a differential operator, has a unique bounded smooth solution such that $V(T, \cdot) = g(\cdot)$.]

END OF PAPER