MAT3
MATHEMATICAL TRIPOS Part III

Friday, 2 June, 2023 9:00 am to 12:00 pm

PAPER 209

LATTICE MODELS

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

Attempt **ALL** questions.
There are **FOUR** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 (i) Define the $O(2)$ model (with free boundary conditions) on a finite graph $\Lambda \subset \mathbb{Z}^d$ and state and prove the Ginibre inequality.

(ii) Show that the magnetisation $\langle \sigma_x^1 \rangle_{\beta,h}^{\Lambda}$ is monotone in the external field $h \geqslant 0$ (pointing in the first coordinate direction).

(iii) Assume that the external field is 0. Show that the infinite volume limit of the expectation $\langle \cos(2\theta_x) - \sin(\theta_y) \rangle_{\beta,0}^{\Lambda}$ exists for any $x, y \in \mathbb{Z}^d$.

(iv) Let $\bar{\Lambda}$ be the closure of Λ, i.e., $\{x \in \mathbb{Z}^d : \operatorname{dist}(x, \Lambda) \leqslant 1\}$ and let $\bar{E}$ denote the corresponding edges, i.e., the edges of $\mathbb{Z}^d$ for which at least one endpoint is in Λ. For each finite square $\Lambda \subset \mathbb{Z}^2$ approximately centered at 0, find $\varphi \in \mathbb{R}^{\bar{\Lambda}}$ with $\varphi_0 = 1$ and $\varphi|_{\partial\Lambda} = 0$ (where $\partial\Lambda = \bar{\Lambda} \setminus \Lambda$) such that $\sum_{xy \subset \bar{E}} (\varphi_x - \varphi_y)^2 \to 0$ as $\Lambda \to \mathbb{Z}^2$.

2 Let Λ_L be the discrete d-dimensional torus with an even number of vertices L in every coordinate direction.

(i) State the Gaussian domination bound for the $O(n)$ model on Λ.

(ii) Prove that when $d \geqslant 3$ there exists $\beta_0 \in (0, \infty)$ such that $\liminf_{h \downarrow 0} \liminf_{L \to \infty} \langle \sigma_0 \cdot e \rangle_{\beta,h}^{\Lambda_L} \geqslant c > 0$ for $\beta > \beta_0$. Here $e \in \mathbb{R}^n$ denotes the (unit) direction of the magnetic field. What happens if the limit $\liminf_{L \to \infty} \liminf_{h \downarrow 0} \langle \sigma_0 \cdot e \rangle_{\beta,h}^{\Lambda_L}$? Explain briefly.

iii) Assuming that the limit $m = \lim_{h \downarrow 0} \lim_{L \to \infty} \langle \sigma_0 \cdot e \rangle_{\beta,h}^{\Lambda_L}$ exists, find the limit $\lim_{h \downarrow 0} \lim_{L \to \infty} \langle \sigma_0 \cdot u \rangle_{\beta,h}^{\Lambda_L}$ for any $u \in \mathbb{R}^n$.

(iv) Let μ be the uniform probability measure on $\{\pm 1\}$. Show that $\mu^{\otimes \Lambda_L}$ is reflection positive (through sites) and also through edges when L is odd.

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(i) For the Ising model on $\mathbb{Z}$ with free boundary conditions, show that for any $\beta > 0$ there is $c(\beta) > 0$ such that $\langle \sigma_0 \sigma_x \rangle_{\beta,0}^{\mathbb{Z}} \leqslant e^{-c(\beta)|x|}$.

(ii) For the Ising model on $\mathbb{Z}$ with plus or minus boundary conditions, show that $\langle \sigma_0 \rangle_{\beta,0}^{\mathbb{Z},\pm} = 0$ for all $\beta > 0$.

(iii) Compute $\langle \sigma_0 \sigma_1 \cdots \sigma_{2022} \rangle_{\beta,0}^{\mathbb{Z},+}$. [You may use results established in the lectures if you cite them carefully.]

(iv) Show that $\langle \sigma_{x_1} \sigma_{x_2} \sigma_{x_3} \sigma_{x_4} \rangle_{\beta,0}^{\mathbb{Z}} \to 0$ as $\min\{|x_i - x_j| : i \neq j\} \to \infty$. [You may use results established in the lectures if you cite them carefully.]

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(a) (i) Let $\Lambda_L \subset \mathbb{Z}^2$ be a square of side length L (approximately) centred at $0 \in \mathbb{Z}^2$, and consider the Ising model on Λ_L with plus boundary conditions outside Λ_L. Show that $\langle \sigma_0 \rangle_{\beta,0}^{\Lambda_L,+}$ is uniformly bounded below when β is sufficiently large.

(ii) Explain how the limit $\langle \sigma_0 \rangle_{\beta,0}^{\mathbb{Z}^2,+}$ is defined and deduce that $\langle \sigma_0 \rangle_{\beta,0}^{\mathbb{Z}^2,+} > 0$ for β sufficiently large.

(iii) Show that $\langle \sigma_0 \rangle_{\beta,0}^{\mathbb{Z}^3,+} > 0$ for β sufficiently large.

(b) Consider the φ^4 model on $\Lambda \subset \mathbb{Z}^d$ whose expectation is given by

$$\langle F \rangle \propto \int_{\mathbb{R}^\Lambda} e^{-\frac{1}{2}\sum_{xy}(\varphi_x-\varphi_y)^2 - \sum_{x\in\Lambda}\frac{1}{4}g\varphi_x^4 - \sum_{x\in\Lambda}\frac{1}{2}\nu\varphi_x^2}\, F(\varphi)\, d\varphi,$$

where $g > 0$ and $\nu \in \mathbb{R}$. Show that if $\nu > 0$ then $\langle \varphi_x \varphi_y \rangle$ decays exponentially. [You may use the random walk representation of the φ^4 model and any results about the decay of the Green's function of a simple random walk on $\Lambda \subset \mathbb{Z}^2$ without proof.]

END OF PAPER