UNIVERSITY OF CAMBRIDGE

MAMA/208, NST3AS/208, MAAS/208

MAT3

MATHEMATICAL TRIPOS Part III

Thursday, 8 June, 2023 9:00 am to 11:00 am

PAPER 208

CONCENTRATION INEQUALITIES

Before you begin please read these instructions carefully

Candidates have TWO HOURS to complete the written examination.

Attempt no more than **THREE** questions.
There are **FOUR** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 A random variable X with $\mathbb{E}[X] = 0$ is *sub-exponential* if there are nonnegative parameters (ν, α) such that

$$\mathbb{E}\left[e^{\lambda X}\right] \leqslant e^{\lambda^2 \nu/2} \quad \text{for all} \quad |\lambda| < \frac{1}{\alpha}.$$

(a) Show that if X is sub-Gaussian with variance parameter ν, then X is sub-exponential with parameters (ν, α) for all $\alpha > 0$. If $X = Z^2 - 1$ for $Z \sim \mathcal{N}(0,1)$, show that X is not sub-Gaussian for any variance parameter ν, but X is $(4,4)$ sub-exponential. You may use, without proof, the inequality $\frac{e^{-t}}{\sqrt{1-2t}} \leqslant e^{2t^2}$ for all $|t| < 1/4$.

(b) Suppose X is sub-exponential with parameters (ν, α). Show that

$$\mathbb{P}(X \geqslant t) \leqslant \begin{cases} e^{-\frac{t^2}{2\nu}} & \text{if } 0 < t \leqslant \frac{\nu}{\alpha}, \\ e^{-\frac{t}{2\alpha}} & \text{if } t > \frac{\nu}{\alpha}. \end{cases}$$

(c) Suppose $\{X_i\}_{i=1}^n$ are independent random variables such that X_i is sub-exponential with parameters (ν_i, α_i). Identify (ν, α) (in terms of $\{(\nu_i, \alpha_i)\}_{i=1}^n$) such that $\sum_{i=1}^n X_i$ is sub-exponential with parameters (ν, α).

(d) A random variable X with $\mathbb{E}[X] = 0$ and $\mathbb{E}[X^2] = \nu$ is said to satisfy *Bernstein's condition* with parameter b if

$$|\mathbb{E}[X^k]| \leqslant \frac{1}{2} k! \nu b^{k-2} \quad \text{for all integers} \quad k \geqslant 2.$$

Show that if X satisfies Bernstein's condition with parameter b, then X is sub-exponential with parameters $(2\nu, 2b)$. You may use, without proof, the inequality $1 + x \leqslant e^x$ for $x \in \mathbb{R}$.

2 A probability distribution p on $\mathbb{R}^n$ is said to satisfy a c-Poincaré inequality if, for $X \sim p$, the following inequality holds for all continuously differentiable functions $f : \mathbb{R}^n \to \mathbb{R}$:

$$\mathrm{Var}(f(X)) \leqslant c^2 \mathbb{E}[\|\nabla f(X)\|^2].$$

(a) Show that the one-dimensional Gaussian $\mathcal{N}(\mu, \sigma^2)$ satisfies a σ-Poincaré inequality.

(b) Suppose $\{p_i\}_{i=1}^N$ are probability distributions on $\mathbb{R}^n$ such that p_i satisfies a c_i-Poincaré inequality. Show that the product distribution $p_1 \otimes p_2 \otimes \cdots \otimes p_N$ satisfies a c-Poincaré inequality with $c = \max\{c_1, c_2, \ldots, c_n\}$.

(c) Let p be a distribution on $\mathbb{R}^n$ satisfying a c-Poincaré inequality. Let $\phi : \mathbb{R}^n \to \mathbb{R}$ be a continuously differentiable function satisfying $\|\nabla \phi(x)\| \leqslant L$ for all $x \in \mathbb{R}^n$. If $X \sim p$, let the distribution of $Y = \phi(X)$ be denoted by q. Show that q satisfies a cL-Poincaré inequality.

(d) Let p be the uniform distribution on $[0, 1]^n$. Find a constant c such that p satisfies a c-Poincaré inequality.

You may use any results from the lectures, provided you state them clearly.

3

(a) State and prove the Modified Log-Sobolev Inequality (MLSI). You may use the tensorization property of entropy without proof, and the fact that for a random variable $Y \geqslant 0$,

$$\mathrm{Ent}(Y) = \inf_{u>0} \mathbb{E}\left[Y(\log Y - \log u) - (Y - u)\right].$$

(b) A non-negative function $f : \mathcal{X}^n \to \mathbb{R}$ is called *weakly-self-bounding* if there exist functions $f_i : \mathcal{X}^{n-1} \to \mathbb{R}$ such that for all $x \in \mathcal{X}^n$,

$$\sum_{i=1}^{n} \Big(f(x) - f_i(x^{(i)}) \Big)^2 \leqslant f(x),$$

and for all $i = 1, \ldots, n$,

$$f_i(x^{(i)}) \leqslant f(x) \quad \text{for all } x \in \mathcal{X}^n.$$

Suppose $Z = f(X_1, \ldots, X_n)$, where $X_1, \ldots, X_n$ are independent random variables on $\mathcal{X}$ and f is a weakly-self-bounding function. Show that for $0 \leqslant \lambda < 2$,

$$\log \mathbb{E}\Big[e^{\lambda(Z - \mathbb{E}Z)}\Big] \leqslant \frac{\lambda^2 \mathbb{E}[Z]}{(2 - \lambda)}.$$

You may use, without proof, the inequality $\phi(-x) \leqslant x^2/2$ for $x \geqslant 0$, where $\phi(t) = e^t - t - 1$. *[Hint: Use the MLSI and rewrite the resulting inequality in terms of $\psi(\lambda) = \log \mathbb{E} e^{\lambda(Z - \mathbb{E}Z)}$ and its derivative $\psi'(\lambda)$.]*

4 Consider a vector $x = (x_1, \ldots, x_n)$ of n distinct numbers in $[0,1]$. The positive integers $1 \leqslant i_1 < i_2 < \cdots < i_m \leqslant n$ form an *increasing sub-sequence* if $x_{i_1} \leqslant x_{i_2} \leqslant \ldots \leqslant x_{i_m}$. Let $L(x)$ denote the length of the longest increasing sub-sequence.

Let $X_1, \ldots, X_n$ be independent random variables supported on $[0,1]$ and let $Z = L(X)$, where $X = (X_1, \ldots, X_n)$.

(a) Show that $\mathrm{Var}(Z) \leqslant n/4$.

(b) Show that for $t \geqslant 0$,

$$\mathbb{P}(Z - \mathbb{E}Z > t) \leqslant e^{-\frac{2t^2}{n}}, \quad \text{and}$$
$$\mathbb{P}(Z - \mathbb{E}Z < -t) \leqslant e^{-\frac{2t^2}{n}}.$$

(c) Show that $\mathrm{Var}(Z) \leqslant \mathbb{E}[Z]$.

(d) Prove that for $t \geqslant 0$,

$$\mathbb{P}(Z - \mathbb{E}Z < -t) \leqslant e^{-\frac{t^2}{2\mathbb{E}[Z]}}.$$

You may use any results from the lectures, provided you state them clearly.

END OF PAPER